Put Options with Linear Investment for Hull-White Interest Rates

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Abstract

We derive a Put Option price associated with selling strategy of the underlying security in a random interest rate environment. This extends Put Option pricing under linear investment strategy from the Black-Scholes setting to Hull-White stochastic interest rate model. As an application, Call Option price for the linear investment strategy in the Hull-White model is established. Our results address recent emergence of developing dynamic investment strategies for the purpose of reducing the investor risk exposure associated with European-type options.

Keywords

European Put Option, Linear Stock Investment Strategy, Zero-Coupon Bond, Change of Numeraire, T-Forward Measure, Hull-White Model

1. Introduction

A steady growth of financial derivatives market over the past decades led to various generalizations of the classical Black-Scholes model. Notably, a pioneering work of Wang *et al.* ([1] [2] [3]) introduced a dynamic investing strategy in the underlying security for the purpose of hedging the investment risks. It turned out that selling a security proportionally to its dropping price for Put Option and buying the security proportionally to its rising price for Call Option (both under the Black-Scholes model) resulted in lower Option price. Zhang *et al.* [4] extended the result for Call Option to stochastic interest rates following the Vasicek model. Subsequently, Ghorbani and Korzeniowski [5] obtained the Call Option price with investment strategy for the Cox-Ingersoll-Ross (CIR) interest rates model via path-integral representation based on n-dimensional Ornstein-Uhlenbeck process. It is worth noting that unlike in the Vasicek model, where the interest rate is gaussian, the interest rate process in the CIR model is no longer gaussian and lacks the closed form representation.

The gist of considering dynamic investment strategies, such as presented here, is two-fold. Firstly, unlike in the classical Black-Scholes model where the investor buys options and has no position in the underlying stock throughout the option time horizon, the dynamic investment strategy requires the investor to continuously trade the stock, whereby lowering the investor risk which is manifested by the lower option price. Secondly, the interest rates are no longer constant and are assumed stochastic.

This paper is concerned with Put Option hedging by linear investment strategy under the Hull-White stochastic interest rates model. European Put Option with the linear investment strategy triggers stock selling whenever the stock price falls below the strike price and stays in the range $\left[(1-\alpha)K, K\right]$. Following [2] we state the relevant facts regarding the hedging strategy. The investment fraction is defined by:

$$Q(S)=\begin{cases}\beta & S\le(1-\alpha)K\\ \dfrac{(1-\beta)}{\alpha K}\left[S-(1-\alpha)K\right]+\beta & (1-\alpha)K\le S\le K\\ 1 & S\ge K\end{cases}\tag{1.1}$$

where

S is stock price.

$Q(S)$ is the stock investment proportion, which is equal to the value of the stock investment divided by A, where A is the entire investment amount.

K is strike price of the option.

α is the investment strategy index, indicating the stock investment occurs during the period in which the stock price drops from K to $(1-\alpha)K$.

β is the minimum value of the stock investment proportion.

It was found in [2] that the Put Option value V_T based on the linear investment with parameters α,β, strike price K reads as follows:

$$V_T=\begin{cases}0 & S_T\ge K\\ \dfrac{1-\beta}{\alpha}\left(\dfrac{2\alpha-1}{2}K+(1-\alpha)S_T-\dfrac{S_T^2}{2K}\right) & (1-\alpha)K\le S_T\le K\\ (K-\beta S_T)-\dfrac{(1-\beta)K(2-\alpha)}{2} & S_T\le(1-\alpha)K\end{cases}\tag{1.2}$$

where S_T is the terminal stock price.

We will use the above formula for the stock price that satisfies a stochastic differential equation (*SDE*) with drift depending on the random interest rate, whose *SDE* follows the Hull-White model.

2. The Market Model

The evolution of the stock price S_t satisfies the following *SDE*

$$\mathrm{d}S_t=\mu_t S_t\mathrm{d}t+\sigma_1 S_t\mathrm{d}W_{1,t}\tag{2.1}$$

with mean return μ_t, constant volatility σ_1 and a standard Brownian motion $W_{1,t}$. The stock price dynamic under the risk-neutral measure is then as follows

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_1 S_t \mathrm{d}W_{1,t} \tag{2.2}$$

where r_t is the interest rate.

By Ito formula the stock price at time T can be expressed as

$$S_T = S_0 \mathrm{e}^{\int_0^T \left(r_s - \frac{\sigma_1^2}{2} \right) \mathrm{d}s + \int_0^T \sigma_1 \mathrm{d}W_{1,t}} \tag{2.3}$$

where S_0 is the initial stock price.

Wang *et al.* [2] proposed a put option model based on a dynamic investment strategy for the Black-Scholes option pricing. In this paper we extend their result to the stochastic Hull-White interest rate model r_t which satisfies the following *SDE*

$$\mathrm{d}r_t = \left(\theta(t) - a r_t \right) \mathrm{d}t + \sigma_2 \mathrm{d}W_{2,t} \tag{2.4}$$

with a and σ_2 constants and $W_{2,t}$ standard Brownian motion independent from $W_{1,t}$.

Remark 2.1. Special case of Hull-White model, $\theta(t) = ab$, becomes the Vasicek model.

In general, Calin [6], $\theta(t)$ satisfies the following equation

$$\theta(t) = \partial_t f(0,t) + a f(0,t) + \frac{\sigma^2}{2a}\left(1 - \mathrm{e}^{-2at}\right) \tag{2.5}$$

where $f(0,t)$ is the yield curve determined by the bond price.

The solution to (2.4) reads

$$r_t = r_0 \mathrm{e}^{-at} + \mathrm{e}^{-at} \int_0^t \theta(s) \mathrm{e}^{as} \mathrm{d}s + \sigma_2 \mathrm{e}^{-at} \int_0^t \mathrm{e}^{as} \mathrm{d}W_{2,s}. \tag{2.6}$$

Note that the first two terms are deterministic and the last is a Wiener integral, thus the process r_t is normally distributed, with mean and variance

$$\begin{aligned} E[r_t] &= r_0 \mathrm{e}^{-at} + \mathrm{e}^{-at} \int_0^t \theta(s) \mathrm{e}^{as} \mathrm{d}s \\ Var[r_t] &= \frac{\sigma_2^2}{2a}\left(1 - \mathrm{e}^{-2at}\right) \end{aligned} \tag{2.7}$$

Integrating (2.6) yields

$$\int_0^t r_s \mathrm{d}s = \frac{r_0\left(1 - \mathrm{e}^{-at}\right)}{a} + \int_0^t \mathrm{e}^{-as} \int_0^s \theta(u) \mathrm{e}^{au} \mathrm{d}u \mathrm{d}s + \sigma_2 \int_0^t \mathrm{e}^{-as} \int_0^s \mathrm{e}^{au} \mathrm{d}W_{2,u} \tag{2.8}$$

when the interest rates are stochastic, the bond price is calculated by conditional expectation

$$P(t,T) = E\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \,|\, \mathcal{F}_t \right] \tag{2.9}$$

where $\mathcal{F}_t$ denotes the information available in the market at time *t*.

Lemma 2.1. The Hull-White zero-coupon bond price is as follows

$$P(0,T) = \mathrm{e}^{\frac{r_0\left(\mathrm{e}^{-aT} - 1\right)}{a} - \int_0^T \mathrm{e}^{-as} \int_0^s \theta(u) \mathrm{e}^{au} \mathrm{d}u \mathrm{d}s + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1 - \mathrm{e}^{-2aT}}{2a} - \frac{2}{a}\left(1 - \mathrm{e}^{-aT}\right) \right]} \tag{2.10}$$

Proof: By (2.9)

$$\begin{aligned} P(0,T) &= E\left[\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} \middle| \mathcal{F}_0\right] \\ &= \mathrm{e}^{\frac{r_0\left(1-\mathrm{e}^{-aT}\right)}{a}} \mathrm{e}^{-\int_0^T \mathrm{e}^{-as}\int_0^s \theta(u)\mathrm{e}^{au}\mathrm{d}u\mathrm{d}s} E\left[\mathrm{e}^{-\sigma_2\int_0^T \mathrm{e}^{-as}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\mathrm{d}s} \middle| \mathcal{F}_0\right] \end{aligned} \tag{2.11}$$

The proof will be carried out in several steps.

Step 1: Set $X_T = \int_0^T \mathrm{e}^{-as}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\mathrm{d}s$, then

$$E[X_T] = \int_o^T \mathrm{e}^{-as} E\left[\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\right]\mathrm{d}s = 0 \tag{2.12}$$

since $\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}$ is gaussian with mean 0 and variance $\frac{\mathrm{e}^{2as}-1}{2a}$.

Step 2:

By product rule

$$\begin{aligned} \mathrm{d}\left(X_s\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\right) &= \int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\mathrm{d}X_s + X_s\mathrm{d}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u} + \underbrace{\mathrm{d}X_s\mathrm{d}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}}_{0} \\ &= \mathrm{e}^{-as}\left(\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\right)^2\mathrm{d}s + X_s\mathrm{e}^{as}\mathrm{d}W_{2,s} \end{aligned} \tag{2.13}$$

Integrating the above gives

$$X_T\int_0^T \mathrm{e}^{as}\mathrm{d}W_{2,s} = \int_0^T \mathrm{e}^{-as}\left(\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\right)^2\mathrm{d}s + \int_0^T X_s\mathrm{e}^{as}\mathrm{d}W_{2,s} \tag{2.14}$$

By taking the expectation and using the fact that the Wiener integral has zero mean, we obtain

$$\begin{aligned} E\left[X_T\int_0^T \mathrm{e}^{as}\mathrm{d}W_{2,s}\right] &= \int_0^T \mathrm{e}^{-as}E\left[\left(\int_0^T \mathrm{e}^{as}\mathrm{d}W_{2,s}\right)^2\right]\mathrm{d}s \\ &= \int_0^T \mathrm{e}^{-as}\left(\frac{\mathrm{e}^{2as}-1}{2a}\right)\mathrm{d}s \\ &= \frac{1}{a^2}\left(\frac{\mathrm{e}^{aT}+\mathrm{e}^{-aT}}{2}-1\right) \end{aligned} \tag{2.15}$$

Step 3: Applying Ito lemma:

$$\mathrm{d}\left(X_T^2\right) = 2X_T\mathrm{d}X_T + \left(\mathrm{d}X_T\right)^2 = 2X_T\mathrm{e}^{-aT}\int_0^T \mathrm{e}^{as}\mathrm{d}W_{2,s}\mathrm{d}t \tag{2.16}$$

then integrating and applying step 2, yields

$$\begin{aligned} E\left[X_T^2\right] &= 2\int_0^T \mathrm{e}^{-as}E\left[X_s\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\right]\mathrm{d}s = \frac{2}{a^2}\int_0^T\left(\frac{1+\mathrm{e}^{-2as}}{2}-\mathrm{e}^{-as}\right)\mathrm{d}s \\ &= \frac{1}{a^2}\left[T+\frac{1}{2a}\left(1-\mathrm{e}^{-2aT}\right)+\frac{2}{a}\left(\mathrm{e}^{-aT}-1\right)\right] \end{aligned} \tag{2.17}$$

Step 4: Using a stochastic variant of Fubini's theorem we interchange the Riemann and the Wiener integrals as follows

$$\begin{aligned} X_T &= \int_0^T \mathrm{e}^{-as}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\mathrm{d}s = \int_0^s \mathrm{e}^{au}\int_0^T \mathrm{e}^{-as}\mathrm{d}s\mathrm{d}W_{2,u} \\ &= \int_0^T \mathrm{e}^{-as}\mathrm{d}s\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u} = \frac{1}{a}\left(1-\mathrm{e}^{-aT}\right)\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u} \end{aligned} \tag{2.18}$$

which implies that X_T is normally distributed with mean 0 and variance $E\left[X_T^2\right]$ computed in step 3. Therefore

$$E\left[\mathrm{e}^{-\sigma_2\int_0^T \mathrm{e}^{-as}\int_0^s \mathrm{e}^{au}\mathrm{d}W_{2,u}\mathrm{d}s}\right]=E\left[\mathrm{e}^{\sigma_2 X_T}\right]=\mathrm{e}^{\left(\frac{\sigma_2^2}{2}\right)Var(X_T)}=\mathrm{e}^{\frac{\sigma_2^2}{2a^2}\left[T+\frac{1-\mathrm{e}^{-2aT}}{2a}-\frac{2}{a}\left(1-\mathrm{e}^{-aT}\right)\right]} \tag{2.19}$$

which gives rise to the formula (zero coupon bond price).

3. Hull-White under T-Forward Measure

The stochastic model for the bond price under Hull-White model is as follows [6]

$$\begin{aligned}\mathrm{d}P(t,T)&=r_tP(t,T)\mathrm{d}t+v(t,T)P(t,T)\mathrm{d}W_t\\&\equiv r_tP(t,T)\mathrm{d}t-\frac{\sigma_2}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)P(t,T)\mathrm{d}W_t\end{aligned} \tag{3.1}$$

In order to simplify the calculation of option value under the stochastic interest rate, we use the technique of changing the measure and numeraire. Following general considerations in Brigo and Mercurio [7] the dynamic of Hull-White model under the zero-coupon bond as numeraire can be obtained using the following

Proposition 2.3.1. [7] Assume two numeraires B and P evolve under a probability measure Q

$$\begin{aligned}\mathrm{d}B_t&=(\ldots)\mathrm{d}t+\sigma_t^B\mathrm{d}W_t\\\mathrm{d}P_t&=(\ldots)\mathrm{d}t+\sigma_t^P\mathrm{d}W_t\end{aligned} \tag{3.2}$$

Then the drift of process X under numeraire P is

$$\mu_t^P(X_t)=\mu_t^B(X_t)-\sigma_t(X_t)\left(\frac{\sigma_t^B}{B_t}-\frac{\sigma_t^P}{P_t}\right) \tag{3.3}$$

and

$$\mathrm{d}W_t^P=\mathrm{d}W_t+\left(\frac{\sigma_t^B}{B_t}-\frac{\sigma_t^P}{P_t}\right)\mathrm{d}t. \tag{3.4}$$

Note. By the Proposition for money market account $\mathrm{d}B_t=r_tB_t\mathrm{d}t$ and zero-coupon bond $\mathrm{d}P(t,T)=r_tP(t,T)\mathrm{d}t-\frac{\sigma}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)P(t,T)\mathrm{d}W_{2,t}$, r_t for Hull-White model under T-forward measure Q^T satisfies the following *SDE*

$$\mathrm{d}r_t=\left(\theta(t)-ar_t-\frac{\sigma_2^2}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)\right)\mathrm{d}t+\sigma\mathrm{d}W_t^T \tag{3.5}$$

where

$$\mathrm{d}W_t^T=\mathrm{d}W_t+\frac{\sigma}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)\mathrm{d}t. \tag{3.6}$$

Now that we obtained the evolution of Hull-White under T-forward measure, we solve (3.5) via multiplying by the integrating factor e^{at} to get

$$\mathrm{d}\left(r_t \mathrm{e}^{at}\right) = \mathrm{e}^{at}\theta(t)\mathrm{d}t - \mathrm{e}^{at}\frac{\sigma_2^2}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)\mathrm{d}t + \mathrm{e}^{at}\sigma_2 \mathrm{d}W_{2,t}^T. \tag{3.7}$$

Integrating from 0 and t yields

$$\begin{aligned} r_t &= r_0\mathrm{e}^{-at} - \frac{\sigma_2^2}{a}\left[\frac{\left(\mathrm{e}^{at}-1\right)\mathrm{e}^{-aT}\left(2\mathrm{e}^{aT}-\mathrm{e}^{at}-1\right)}{2a}\right]\mathrm{e}^{-at} \\ &\quad + \mathrm{e}^{-at}\int_0^t \theta(s)\mathrm{e}^{as}\mathrm{d}s + \sigma_2\int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}W_{2,s}^T \end{aligned} \tag{3.8}$$

By integrating over [0, T] we obtain

$$\begin{aligned} \int_0^T r_t\mathrm{d}t &= r_0\frac{1-\mathrm{e}^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{\mathrm{e}^{-2aT}\left((2aT-3)\mathrm{e}^{2aT}+4\mathrm{e}^{aT}-1\right)}{2a^2}\right] \\ &\quad + \int_0^T \mathrm{e}^{-at}\int_0^t \theta(s)\mathrm{e}^{as}\mathrm{d}s\mathrm{d}t + \sigma_2\int_0^T\int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}W_{2,u}^T\mathrm{d}t. \end{aligned} \tag{3.9}$$

4. Put Option Price

The Put Option price is expressed as a product of the expectation of V_T under the T-forward measure and the price of zero-coupon bond. Notice that by (2.3) stock price S_T is lognormally distributed and we denote its probability density by $f(s)$ for $S_T = s$.

Theorem 4.1. (Put Option Price). The Put Option price at time 0 under the Hull-White interest rate is given by

$$\begin{aligned} P_T &= P(0,T)\left(\left(K - \frac{(1-\beta)K(2-\alpha)}{2}\right)N[d_1] - \beta\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}N[d_2]\right) \\ &\quad - P(0,T)\left(\frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\left[N(d_3)-N(d_1)\right]\right) \\ &\quad - P(0,T)\left(\frac{1-\beta}{\alpha}(1-\alpha)\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N(d_4)-N(d_2)\right]\right) \\ &\quad + P(0,T)\left(\frac{\beta-1}{2\alpha K}\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N(d_4)-N(d_2)\right]\right) \end{aligned} \tag{4.1}$$

where

$$P(0,T) = \mathrm{e}^{\frac{r_0\left(\mathrm{e}^{-aT}-1\right)}{a} - \int_0^T \mathrm{e}^{-as}\int_0^s \theta(u)\mathrm{e}^{au}\mathrm{d}u\mathrm{d}s + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1-\mathrm{e}^{-2aT}}{2a} - \frac{2}{a}\left(1-\mathrm{e}^{-aT}\right)\right]}$$

$$d_1 = \frac{\ln(1-\alpha)K - \mu_T}{\sigma_T} \qquad d_2 = \frac{\ln(1-\alpha)K - \mu_T - \sigma_T^2}{\sigma_T}$$

$$d_3 = \frac{\ln K - \mu_T}{\sigma_T} \qquad d_4 = \frac{\ln K - \mu_T - \sigma_T^2}{\sigma_T}$$

$$\begin{aligned} \mu_T &= \ln S_0 - \frac{\sigma_1^2}{2}T + r_0\frac{1-\mathrm{e}^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{\mathrm{e}^{-2aT}\left((2aT-3)\mathrm{e}^{2aT}+4\mathrm{e}^{aT}-1\right)}{2a^2}\right] \\ &\quad + \int_0^T \mathrm{e}^{-at}\int_0^t \theta(s)\mathrm{e}^{as}\mathrm{d}s\mathrm{d}t \end{aligned}$$

$$\sigma_T^2 = \sigma_1^2 T + \frac{\sigma^2}{a^2}\left[T - 2\frac{1-\mathrm{e}^{-aT}}{a} + \frac{1-\mathrm{e}^{-2aT}}{2a}\right]$$

Proof.

$$
\begin{aligned}
E^T\left[V_T\right] &= \int_0^{(1-\alpha)K}\left[\left(K-\beta S_T\right)-\frac{(1-\beta)K(2-\alpha)}{2}\right] f\left(S_T\right)\mathrm{d}S_T \\
&\quad +\int_{(1-\alpha)K}^{K}\frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K+(1-\alpha)S_T-\frac{S_T^2}{2K}\right] f\left(S_T\right)\mathrm{d}S_T \\
&= \int_0^{(1-\alpha)K}\left[\left(K-\beta S_T\right)-\frac{(1-\beta)K(2-\alpha)}{2}\right] f\left(S_T\right)\mathrm{d}S_T \\
&\quad +\int_{(1-\alpha)K}^{K}\frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K+(1-\alpha)S_T\right] f\left(S_T\right)\mathrm{d}S_T \\
&\quad -\int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right] f\left(S_T\right)\mathrm{d}S_T
\end{aligned} \tag{4.2}
$$

We split evaluating $E^T\left[V_T\right]$ into integrals I_1, I_2 and I_3 as follows

$$
\begin{aligned}
I_1 &= \int_0^{(1-\alpha)K}\left[\left(K-\beta S_T\right)-\frac{(1-\beta)K(2-\alpha)}{2}\right] f\left(S_T\right)\mathrm{d}S_T \\
I_2 &= \int_{(1-\alpha)K}^{K}\frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K+(1-\alpha)S_T\right] f\left(S_T\right)\mathrm{d}S_T \\
I_3 &= -\int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right] f\left(S_T\right)\mathrm{d}S_T
\end{aligned} \tag{4.3}
$$

Set $y=\ln S_T$, then $f\left(\mathrm{e}^y\right)\mathrm{e}^y$ is the probability density function of $\ln S_T$, and the mean and variance under the T-forward measure can be expressed form (2.3) as follows

$$
\begin{aligned}
\mu_T &= E^T\left[\ln S_T\right] = E^T\left[\ln S_0+\int_0^T\left(r_t-\frac{\sigma_1^2}{2}\right)\mathrm{d}t+\int_0^T\sigma_1\mathrm{d}W_{1,t}^T\right] \\
&= \ln S_0-\frac{\sigma_1^2}{2}T+E^T\left[\int_0^T r_t\mathrm{d}t\right]+\underbrace{E^T\left[\int_0^T\sigma_1\mathrm{d}W_{1,t}^T\right]}_{0} \\
&= \ln S_0-\frac{\sigma_1^2}{2}T+r_0\frac{1-\mathrm{e}^{-aT}}{a}-\frac{\sigma_2^2}{a}\left[\frac{\mathrm{e}^{-2aT}\left(\left(2aT-3\right)\mathrm{e}^{2aT}+4\mathrm{e}^{aT}-1\right)}{2a^2}\right] \\
&\quad +\int_0^T\mathrm{e}^{-at}\int_0^t\theta(s)\mathrm{e}^{as}\mathrm{d}s\mathrm{d}t
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
\sigma_T^2 &= Var^T\left[\ln S_T\right] = Var^T\left[\ln S_0+\int_0^T\left(r_t-\frac{\sigma_1^2}{2}\right)\mathrm{d}t+\int_0^T\sigma_1\mathrm{d}W_{1,t}^T\right] \\
&= Var^T\left[\ln S_0\right]+Var^T\left[\int_0^T\left(r_t-\frac{\sigma_1^2}{2}\right)\mathrm{d}t\right]+Var^T\left[\int_0^T\sigma_1\mathrm{d}W_{1,t}^T\right] \\
&= Var^T\left[\int_0^T\sigma_1\mathrm{d}W_{1,t}^T\right]+Var^T\left[\int_0^T\sigma_2\mathrm{e}^{-at}\int_0^t\mathrm{e}^{as}\mathrm{d}W_{2,s}^T\mathrm{d}t\right] \\
&= \int_0^T\sigma_1^2\mathrm{d}t+Var^T\left[\sigma_2\int_0^T\int_0^t\mathrm{e}^{-a(t-s)}\mathrm{d}W_{2,s}^T\mathrm{d}t\right] \\
&= \sigma_1^2T+\frac{\sigma^2}{a^2}\left[T-2\frac{1-\mathrm{e}^{-aT}}{a}+\frac{1-\mathrm{e}^{-2aT}}{2a}\right].
\end{aligned} \tag{4.5}
$$

Moreover, we have

$$f\left(e^{y}\right)e^{y}=\frac{1}{\sqrt{2\pi}\sigma_T}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}} \tag{4.6}$$

and therefore

$$\begin{aligned}
I_1 &= \int_0^{(1-\alpha)K}\left((K-\beta S_T)-\frac{(1-\beta)K(2-\alpha)}{2}\right)f(S_T)\mathrm{d}S_T \\
&= \int_0^{(1-\alpha)K}\left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)f(S_T)\mathrm{d}S_T-\beta\int_0^{(1-\alpha)K}S_T f(S_T)\mathrm{d}S_T \\
&= \int_{-\infty}^{\ln(1-\alpha)K}\left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)f\left(e^{y}\right)e^{y}\mathrm{d}y-\beta\int_{-\infty}^{\ln(1-\alpha)}e^{y}f\left(e^{y}\right)e^{y}\mathrm{d}y \\
&= \left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)\frac{1}{\sqrt{2\pi}\sigma_T}\int_{-\infty}^{\ln(1-\alpha)K}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}\mathrm{d}y \\
&\quad -\beta\frac{1}{\sqrt{2\pi}\sigma_T}\int_{-\infty}^{\ln(1-\alpha)K}e^{y}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}\mathrm{d}y
\end{aligned}$$

Setting $z=\frac{y-\mu_T}{\sigma_T}$ gives

$$\begin{aligned}
I_1 &= \left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}e^{-\frac{1}{2}z^2}\mathrm{d}z \\
&\quad -\beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}e^{\mu_T+z\sigma_T}e^{-\frac{1}{2}z^2}\mathrm{d}z \\
&= \left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}e^{-\frac{1}{2}z^2}\mathrm{d}z \\
&\quad -\beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}e^{\mu_T+\frac{1}{2}\sigma_T^2}e^{-\frac{1}{2}\left(z-\sigma_T^2\right)}\mathrm{d}z \\
&= \left(K-\frac{(1-\beta)K(2-\alpha)}{2}\right)N\left[\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}\right] \\
&\quad -\beta e^{\mu_T+\frac{1}{2}\sigma_T^2}N\left[\frac{\ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}\right]
\end{aligned} \tag{4.7}$$

where $N(x)$ is the standard normal cumulative distribution function.

Similarly,

$$\begin{aligned}
I_2 &= \int_{(1-\alpha)K}^{K}\frac{1-\beta}{\alpha}\left(\frac{2\alpha-1}{2}K+(1-\alpha)S_T\right)f(S_T)\mathrm{d}S_T \\
&= \int_{(1-\alpha)K}^{K}\left(\frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\right)f(S_T)\mathrm{d}S_T+\frac{1-\beta}{\alpha}(1-\alpha)\int_{(1-\alpha)K}^{K}S_T f(S_T)\mathrm{d}S_T \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\int_{\ln(1-\alpha)K}^{\ln K}f\left(e^{y}\right)e^{y}\mathrm{d}y+\frac{1-\beta}{\alpha}(1-\alpha)\int_{\ln(1-\alpha)K}^{\ln K}e^{y}f\left(e^{y}\right)e^{y}\mathrm{d}y \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\frac{1}{\sqrt{2\pi}\sigma_T}\int_{\ln(1-\alpha)K}^{\ln K}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}\mathrm{d}y \\
&\quad +\frac{1-\beta}{\alpha}(1-\alpha)\frac{1}{\sqrt{2\pi}\sigma_T}\int_{\ln(1-\alpha)K}^{\ln K}e^{y}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{\ln K-\mu_T}{\sigma_T}} \mathrm{e}^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&+\frac{1-\beta}{\alpha}(1-\alpha)\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{\ln K-\mu_T}{\sigma_T}} \mathrm{e}^{\mu_T+z\sigma_T}\mathrm{e}^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\left[N\left(\frac{\ln K-\mu_T}{\sigma_T}\right)-N\left(\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}\right)\right] \\
&+\frac{1-\beta}{\alpha}(1-\alpha)\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N\left(\frac{\ln K-\mu_T-\sigma_T^2}{\sigma_T}\right)-N\left(\frac{\ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}\right)\right]
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
I_3 &= -\int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right]f(S_T)\,\mathrm{d}S_T = \frac{\beta-1}{2\alpha K}\int_{(1-\alpha)K}^{K} S_T^2 f(S_T)\,\mathrm{d}S_T \\
&= \frac{\beta-1}{2\alpha K}\int_{\ln(1-\alpha)K}^{\ln K}\mathrm{e}^{2y}f(\mathrm{e}^y)\,\mathrm{d}y = \frac{\beta-1}{2\alpha K}\int_{\ln(1-\alpha)K}^{\ln K}\mathrm{e}^{y}\mathrm{e}^{y}f(\mathrm{e}^y)\,\mathrm{d}y \\
&= \frac{\beta-1}{2\alpha K}\frac{1}{\sqrt{2\pi}\sigma_T}\int_{\ln(1-\alpha)K}^{\ln K}\mathrm{e}^{y}\mathrm{e}^{-\frac{1}{2}\frac{(y-\mu_T^2)}{\sigma_T^2}}\,\mathrm{d}y \\
&= \frac{\beta-1}{2\alpha K}\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{\ln K-\mu_T}{\sigma_T}}\mathrm{e}^{\mu_T+z\sigma_T}\mathrm{e}^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= \frac{\beta-1}{2\alpha K}\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N\left(\frac{\ln K-\mu_T-\sigma_T^2}{\sigma_T}\right)-N\left(\frac{\ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}\right)\right]
\end{aligned}
\tag{4.9}
$$

Corollary. 4.1. (Put Option price for Vasicek Model). Observe that in the special case we recover the Put Option price for the Vasicek model when $\theta(t) = ab$.

5. Application to Call Option

European Call Option under the linear investment strategy triggers stock buying whenever the stock price exceeds the strike price. The investment fraction is defined by:

$$
Q(S) = \begin{cases} 0 & S \le K \\ \frac{\beta}{\alpha K}(S-K) & K \le S \le (1+\alpha)K \\ \beta & S \ge (1+\alpha)K \end{cases}
$$

where

S is stock price.

$Q(S)$ is the stock investment proportion, which is equal to the value of the stock investment divided by A, where A is the entire investment amount.

K is strike price of the option.

α is the investment strategy index, indicating the stock investment occurs during the period in which the stock price increases from K to $(1+\alpha)K$.

β is the maximum value of the stock investment proportion.

Zhang *et al.* [4] derived the Call Option price $C \equiv C_T$ based on the linear investment for the Vasicek interest rate model and we extend their result to the Hull-White model.

Theorem. 5.1. The Call Option price with the linear investment strategy at time 0 for the Hull-White model is given by

$$
\begin{aligned}
C_T = & P(0,T)\left(1+\frac{\beta}{\alpha}\left(1-\mu_T+\ln K-\sigma_T^2\right)\right)\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N(d_1)-N(d_2)\right] \\
& -P(0,T)\left(1+\frac{\beta}{\alpha}\right)K\left[N(d_3)-N(d_4)\right] \\
& -P(0,T)\frac{\beta}{\alpha}\frac{\sigma_T}{\sqrt{2\pi}}\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}\left(\mathrm{e}^{-\frac{d_2^2}{2}}-\mathrm{e}^{-\frac{d_1^2}{2}}\right) \\
& +P(0,T)\left(1-\frac{\beta}{\alpha}\ln(1+\alpha)\right)\mathrm{e}^{\mu_T+\frac{1}{2}\sigma_T^2}N(-d_1) \\
& +P(0,T)K(\beta-1)N(-d_3)
\end{aligned}
$$

with $P(0,T)$, d_1, d_2, d_3, d_4, μ_T and σ_T^2 defined below

$$
P(0,T)=\mathrm{e}^{\frac{r_0\left(\mathrm{e}^{-aT}-1\right)}{a}-\int_0^T \mathrm{e}^{-as}\int_0^s \theta(u)\mathrm{e}^{au}\mathrm{d}u\mathrm{d}s+\frac{\sigma_2^2}{2a^2}\left[T+\frac{1-\mathrm{e}^{-2aT}}{2a}-\frac{2}{a}\left(1-\mathrm{e}^{-aT}\right)\right]}
$$

$$
d_1=\frac{\ln(1+\alpha)K-\mu_T-\sigma_T^2}{\sigma_T} \qquad d_2=\frac{\ln K-\mu_T-\sigma_T^2}{\sigma_T}
$$

$$
d_3=\frac{\ln(1+\alpha)K-\mu_T}{\sigma_T} \qquad d_4=\frac{\ln K-\mu_T}{\sigma_T}
$$

$$
\begin{aligned}
\mu_T = & \ln S_0-\frac{\sigma_1^2}{2}T+r_0\frac{1-\mathrm{e}^{-aT}}{a}-\frac{\sigma_2^2}{a}\left[\frac{\mathrm{e}^{-2aT}\left((2aT-3)\mathrm{e}^{2aT}+4\mathrm{e}^{aT}-1\right)}{2a^2}\right] \\
& +\int_0^T \mathrm{e}^{-at}\int_0^t \theta(s)\mathrm{e}^{as}\mathrm{d}s\mathrm{d}t
\end{aligned}
$$

$$
\sigma_T^2=\sigma_1^2T+\frac{\sigma^2}{a^2}\left[T-2\frac{1-\mathrm{e}^{-aT}}{a}+\frac{1-\mathrm{e}^{-2aT}}{2a}\right]
$$

Proof. The formula for C_T has been derived in Zhang *et al.* [4] for the Vasicek model with explicit dependence on the bond price $P(0,T)$, μ_T and σ_T^2. Since in the Hull-White model the respective bond price $P(0,T)$, μ_T and σ_T^2 have been found in (2.10), (4.4) and (4.5) respectively, and the derivation of the Call Option price C_T in Hull-White model is analogous to that of Vasicek model we omit the proof of the formula C_T.

Corollary. 5.1. (Call Option price for Vasicek Model). Observe that in the special case we recover the Call Option price for the Vasicek model when $\theta(t)=ab$.

6. Conclusion

We obtained the closed form of the Put and Call Option price for the linear investment strategy under the Hull-White stochastic interest rates. In particular, a protective put option can serve as an insurance policy against losses for the stock holder. Since the option price associated with trading of the underlying security is based on continuous stock trading (impossible to implement!), a feasible discrete variant is in order. Recently Li *et al.* [8] proposed a discretized method for the Call Option under the classical Black-Scholes with linear investment strategy. A feasible market implementation for our Hull-White pricing model will be presented in a forthcoming paper. Regarding the subject of dynamic investment strategies for European-type options under stochastic interest rates, to the best of our knowledge, the references cited in this article include up to date published research.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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