Trigonometry

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July 29, 2021

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About this Printing of the Textbook

This printing of *Trigonometry* is almost identical to previous printings of the book. Several typographical errors have been corrected, and Sections 3.6 and 5.1 have been reorganized. In addition, the figure numbers for Chapter 3 after Section 3.1 have changed. The pagination for this printing is identical to previous printings until Section 3.6.

License

Cover Photograph: This photograph was taken at Zion National Park on October 5, 2015. The phases of the moon follow a cyclic pattern that can be modeled by trigonometric functions. See Exercise 8 on page 128.

Contents

Note to Students

This book may be different than other mathematics textbooks you have used in the past. In this book, the reader is expected to do more than read the book and is expected to study the material in the book by working out examples rather than just reading about them. So this book is not just about mathematical content but is also about the process of learning and doing mathematics. Along the way, you will also learn some important mathematical topics that will help you in your future study of mathematics.

This book is designed not to be just casually read but rather to be *engaged*. It may seem like a cliché (because it is in almost every mathematics book now) but there is truth in the statement that *mathematics is not a spectator sport*. To learn and understand mathematics, you must *engage* in the process of doing mathematics. So you must actively read and study the book, which means to have a pencil and paper with you and be willing to follow along and fill in missing details. This type of engagement is not easy and is often frustrating, but if you do so, you will learn a great deal about mathematics and more importantly, about doing mathematics.

Recognizing that actively studying a mathematics book is often not easy, several features of the textbook have been designed to help you become more engaged as you study the material. Some of the features are:

- **Beginning Activities.** The introductory material in almost every section of this book contains a so-called beginning activity. Some beginning activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other beginning activities will introduce new concepts and definitions that will be used later in that section. It is very important that you work on these beginning activities before starting the rest of the section. Please note that answers to these beginning activities are not included in the text, but the answers will be developed in the material later in that section.

- **Focus Questions.** At the start of each section, we list some focus questions that provide information about what is important and what ideas are the main focus of the section. A good goal for studying section is to be able answer each of the focus questions.

- **Progress Checks.** Several Progress Checks are included in each section. These are either short exercises or short activities designed to help you determine if you are understanding the material as you are studying the material in the section. As such, it is important to work through these progress checks to test your understanding, and if necessary, study the material again before proceeding further. So it is important to attempt these progress checks before checking the answers, which are provided in Appendix A.

- **Section Summaries.** To assist you with studying the material in the text, there is a summary at the end of each of the sections. The summaries usually list the important definitions introduced in the section and the important results proven in the section. In addition, although not given in a list, the section summaries often contain answers to the focus questions given at the beginning of the section.

- **Answers for Selected Exercises.** Answers or hints for several exercises are included in an Appendix B. Those exercises with an answer or a hint in the appendix are preceded by a star (⋆).

- **Interactive Geogebra Applets.** The text contains links to several interactive Geogebra applets or worksheets. These are active links in the pdf version of the textbook, so clicking on the link will take you directly to the applet. Short URL's for these links have been created so that they are easier to enter if you are using a printed copy of the textbook.

  Following is a link to the GVSU MTH 123 playlist of Geogebra applets on the Geogebra website. (MTH 123 is the trigonometry course at Grand Valley State University.)

  http://gvsu.edu/s/0v

  These applets are usually part of a beginning activity or a progress check and are intended to be used as part of the textbook. See page 15 for an example of a link to an applet on the Geogebra website. This one is part of Progress Check 1.6 and is intended to reinforce the unit circle definitions of the cosine and sine functions.

- **Video Screencasts.** Although not part of the textbook, there are several online videos (on YouTube) that can be used in conjunction with this textbook. There are two sources for video screencasts.

  1. The MTH 123 Playlist on Grand Valley's Department of Mathematics YouTube channel:

  http://gvsu.edu/s/MJ

  Note: MTH 123 is the course number for the trigonometry course at Grand Valley State University.

  2. MTH 123 video screencasts on *Rocket Math 1*. These video screencasts were created by Lynne Mannard, an affiliate faculty member in the Department of Mathematics at Grand Valley State University.

  http://gvsu.edu/s/0cc

- **Website for the Book.** There is a website for the book at

  www.tedsundstrom.com/trigonometry

Preface

This text was written for the three-credit trigonometry course at Grand Valley State University (MTH 123 – Trigonometry). It begins with a circular function approach to trigonometry and transitions to the study of triangle trigonometry, vectors, trigonometric identities, and complex numbers.

The authors are very interested in constructive criticism of the textbook from the users of the book, especially students who are using or have used the book. Please send any comments you have to

trigtext@gmail.com

Important Features of the Textbook

This book is meant to be used and studied by students and the important features of the textbook were designed with that in mind. Please see the *Note to Students* on page (v) for a description of these features.

Content and Organization

The first two chapters of the textbook emphasize the development of the cosine and sine functions and how they can be used to model periodic phenomena. The other four trigonometric functions are studied in Section 1.6 and Section 2.4. Triangles and vectors are studied in Chapter 3, trigonometric identities and equations are studied in Chapter 4, and finally, using trigonometry to better understand complex numbers is in Chapter 5. Following is a more detailed description of the sections within each chapter.

Chapter 1 – The Trigonometric Functions

Section 1.1 introduces the unit circle and the wrapping function for the unit circle. This develops the important relationship between the real numbers and points on the unit circle, which leads to the idea of associating intervals of real numbers with arcs on the unit circle. This is necessary for the development of the cosine and sine functions in Section 1.2. Understanding the ideas in this section is critical for proceeding further in the textbook.

The next two sections are intended to provide a rationale as to why we use radian measure in the development of the trigonometric functions. In addition, calculators and graphing devices are ubiquitous in the study of mathematics now, and when we use a calculator, we need to set the angle mode to radians. One of the purposes of Section 1.3 is to explain why we set our calculators to radian mode. It seems somewhat intellectually dishonest to simply tell students that they must use radian mode and provide no explanation as to why. Section 1.4 can be considered an optional section since it is not used later in the textbook. However, it does provide interesting applications of the use of radian measure when working with linear and angular velocity.

The common arcs $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are introduced in Section 1.5. The exact values of the cosine and sine functions for these arcs are determined using information about the right triangle with two 45° angles and the right triangle with angles of 30° and 60°. An alternate development of these results using points on the unit circle and the distance formula is given in Exercises (9) and (10). Section 1.5 concludes with a discussion of the use of reference arcs, and Section 1.6 introduces the tangent, secant, cosecant, and cotangent functions.

Chapter 2 – Graphs of the Trigonometric Functions

The first three sections of this chapter deal with the graphs of sinusoidal functions and their use in modeling periodic phenomena. The graphs of the cosine and sine functions are developed in Section 2.1 using the unit circle. Geogebra applets are used in this development. Section 2.2 deals with the graphs of sinusoidal functions of the form $y = A\sin(B(x - C)) + D$ or $y = A\cos(B(x - C)) + D$. In this section, it is emphasized that the amplitude, period, and vertical shift for a sinusoidal function is independent of whether a sine or cosine is used. The difference in using a sine or cosine will be the phase shift. Sinusoidal models of periodic phenomena are discussed in Section 2.3. With the use of technology, it is now possible to do sine regressions. Although the textbook is relatively independent of the choice of technology, instructions for doing sine regressions using Geogebra are given in this section.

The graphs of the other four trigonometric functions are developed in Section 2.4. Most of this section can be considered as optional, but it is important to at least discuss the material related to the graph of the tangent function since the inverse tangent function is part of Section 2.5. The inverse sine function and inverse cosine function are, of course, also developed in this section. In order to show how inverse functions can be used in mathematics, solutions of trigonometric equations are studied in Section 2.6.

Chapter 3 – Triangles and Vectors

This chapter contains the usual material dealing with triangle trigonometry including right triangle trigonometry, the Law of Sines, and the Law of Cosines, which are both handled in Section 3.3. The emphasis in this section is how to use these two laws to solve problems involving triangles. By having them both in the same section, students can get practice deciding which law to use for a particular problem. The proofs of these two laws are included as appendices for Section 3.3.

More work with the Law of Cosines and the Law of Sines is included in Section 3.4. In addition, this section contains problems dealing with the area of a triangle including Heron's formula for the area of a triangle. (The proof of Heron's formula is also in an appendix at the end of the section.)

The last two sections of this chapter deal with vectors. Section 3.5 deals with the geometry of vectors, and Section 3.6 deals with vectors from an algebraic point of view.

Chapter 4 – Identities and Equations

The first section of this chapter introduces the concept of a trigonometric identity. The emphasis is on how to verify or prove an identity and how to show that an equation is not an identity. The second section reviews and continues the work on trigonometric equations from Section 2.6.

The last three sections of the chapter cover the usual trigonometric identities in this type of course. In addition, the sections show how identities can be used to help solve equations.

Chapter 5 – Complex Numbers

It is assumed that students have worked with complex numbers before. However, Section 5.1 provides a summary of previous work with complex numbers. In addition, this section introduces the geometric representation of complex numbers in the complex plane. Section 5.2 introduces the trigonometric or polar form of a complex number including the rules for multiplying and dividing complex numbers in trigonometric form. Section 5.3 contains the material dealing with DeMoivre's Theorem about the powers of complex numbers and includes material on how to find roots of complex numbers.

Note to Instructors

Please contact Ted Sundstrom at trigtext@gmail.com for information about an instructors manual with solutions to the beginning activities and the exercises. In your email, please include the name of your institution (school, college, or university), the course for which you are considering using the text, and a link to a website that can be used to verify your position at your institution.

There is also a website for the book at

www.tedsundstrom.com/trigonometry

Chapter 1

The Trigonometric Functions

Trigonometry had its start as the study of triangles. The study of triangles can be traced back to the second millenium B.C.E. in Egyptian and Babylonian mathematics. In fact, the word trigonmetry is derived from a Greek word meaning "triangle measuring." We will study the trigonometry of triangles in Chapter 3. Today, however, the trigonometric functions are used in more ways. In particular, the trigonometric functions can be used to model periodic phenomena such as sound and light waves, the tides, the number of hours of daylight per day at a particular location on earth, and many other phenomena that repeat values in specified time intervals.

Our study of periodic phenomena will begin in Chapter 2, but first we must study the trigonometric functions. To do so, we will use the basic form of a repeating (or periodic) phenomena of travelling around a circle at a constant rate.

1.1 The Unit Circle

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the unit circle and why is it important in trigonometry? What is the equation for the unit circle?
- What is meant by "wrapping the number line around the unit circle?" How is this used to identify real numbers as the lengths of arcs on the unit circle?
- How do we associate an arc on the unit circle with a closed interval of real numbers?

Beginning Activity

As has been indicated, one of the primary reasons we study the trigonometric functions is to be able to model periodic phenomena mathematically. Before we begin our mathematical study of periodic phenomena, here is a little "thought experiment" to consider.

Imagine you are standing at a point on a circle and you begin walking around the circle at a constant rate in the counterclockwise direction. Also assume that it takes you four minutes to walk completely around the circle one time. Now suppose you are at a point P on this circle at a particular time t.

- Describe your position on the circle 2 minutes after the time t.
- Describe your position on the circle 4 minutes after the time t.
- Describe your position on the circle 6 minutes after the time t.
- Describe your position on the circle 8 minutes after the time t.

The idea here is that your position on the circle repeats every 4 minutes. After 2 minutes, you are at a point diametrically opposed from the point you started. After

4 minutes, you are back at your starting point. In fact, you will be back at your starting point after 8 minutes, 12 minutes, 16 minutes, and so on. This is the idea of periodic behavior.

The Unit Circle and the Wrapping Function

In order to model periodic phenomena mathematically, we will need functions that are themselves periodic. In other words, we look for functions whose values repeat in regular and recognizable patterns. Familiar functions like polynomials and exponential functions don't exhibit periodic behavior, so we turn to the trigonometric functions. Before we can define these functions, however, we need a way to introduce periodicity. We do so in a manner similar to the thought experiment, but we also use mathematical objects and equations. The primary tool is something called *the wrapping function*. Instead of using any circle, we will use the so-called **unit circle**. This is the circle whose center is at the origin and whose radius is equal to 1, and the equation for the unit circle is $x^2 + y^2 = 1$.

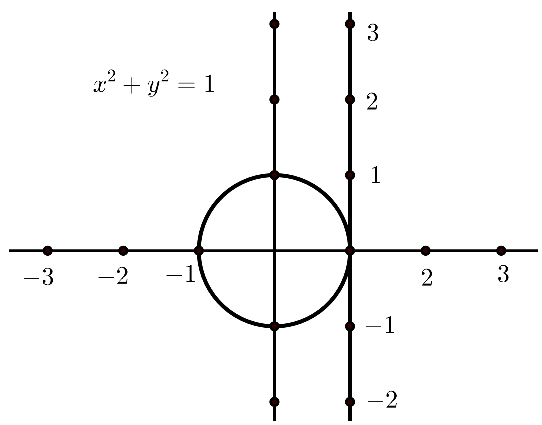


Figure 1.1: Setting up to wrap the number line around the unit circle

Figure 1.1 shows the unit circle with a number line drawn tangent to the circle at the point $(1, 0)$. We will "wrap" this number line around the unit circle. Unlike the number line, the length once around the unit circle is finite. (Remember that the formula for the circumference of a circle as $2\pi r$ where r is the radius, so the length once around the unit circle is 2π.) However, we can still measure distances and locate the points on the number line on the unit circle by wrapping the number line around the circle. We wrap the positive part of this number line around the circumference of the circle in a counterclockwise fashion and wrap the negative part of the number line around the circumference of the unit circle in a clockwise direction.

Two snapshots of an animation of this process for the counterclockwise wrap are shown in Figure 1.2 and two such snapshots are shown in Figure 1.3 for the clockwise wrap.

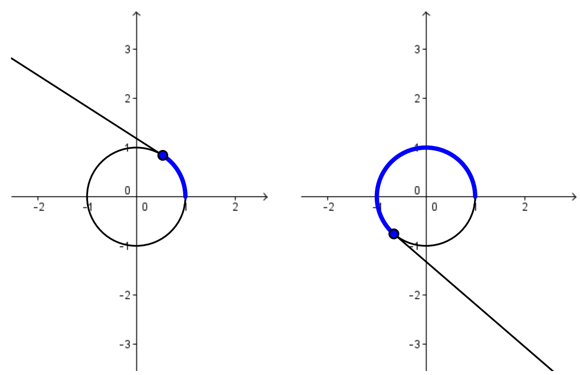


Figure 1.2: Wrapping the positive number line around the unit circle

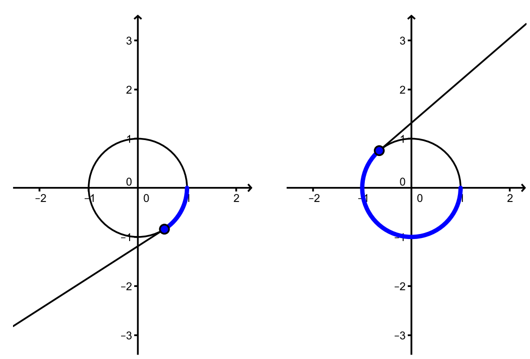


Figure 1.3: Wrapping the negative number line around the unit circle

Following is a link to an actual animation of this process, including both positive wraps and negative wraps.

`http://gvsu.edu/s/Kr`

Figure 1.2 and Figure 1.3 only show a portion of the number line being wrapped around the circle. Since the number line is infinitely long, it will wrap around the circle infinitely many times. A result of this is that infinitely many different numbers from the number line get wrapped to the same location on the unit circle.

- The number 0 and the numbers 2π, -2π, and 4π (as well as others) get wrapped to the point $(1, 0)$. We will usually say that these points get mapped to the point $(1, 0)$.
- The number $\dfrac{\pi}{2}$ is mapped to the point $(0, 1)$. This is because the circumference of the unit circle is 2π and so one-fourth of the circumference is $\dfrac{1}{4}(2\pi) = \dfrac{\pi}{2}$.
- If we now add 2π to $\dfrac{\pi}{2}$, we see that $\dfrac{5\pi}{2}$ also gets mapped to $(0, 1)$. If we subtract 2π from $\dfrac{\pi}{2}$, we see that $-\dfrac{3\pi}{2}$ also gets mapped to $(0, 1)$.

However, the fact that infinitely many different numbers from the number line get wrapped to the same location on the unit circle turns out to be very helpful as it will allow us to model and represent behavior that repeats or is periodic in nature.

Progress Check 1.1 (**The Unit Circle**.)

1. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(-1, 0)$ on the unit circle.
2. Describe all of the numbers on the number line that get wrapped to the point $(-1, 0)$ on the unit circle.
3. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(0, 1)$ on the unit circle.
4. Find two different numbers, one positive and one negative, from the number line that get wrapped to the point $(0, -1)$ on the unit circle.

One thing we should see from our work in Progress Check 1.1 is that integer multiples of π are wrapped either to the point $(1, 0)$ or $(-1, 0)$ and that odd integer multiples of $\dfrac{\pi}{2}$ are wrapped to either to the point $(0, 1)$ or $(0, -1)$. Since the circumference of the unit circle is 2π, it is not surprising that fractional parts of π and the integer multiples of these fractional parts of π can be located on the unit circle. This will be studied in the next progress check.

Progress Check 1.2 (The Unit Circle and π).
The following diagram is a unit circle with 24 equally spaced points plotted on the circle. Since the circumference of the circle is 2π units, the increment between two consecutive points on the circle is $\dfrac{2\pi}{24} = \dfrac{\pi}{12}$.

Label each point with the smallest nonnegative real number t to which it corresponds. For example, the point $(1, 0)$ on the x-axis corresponds to $t = 0$. Moving counterclockwise from this point, the second point corresponds to $\dfrac{2\pi}{12} = \dfrac{\pi}{6}$.

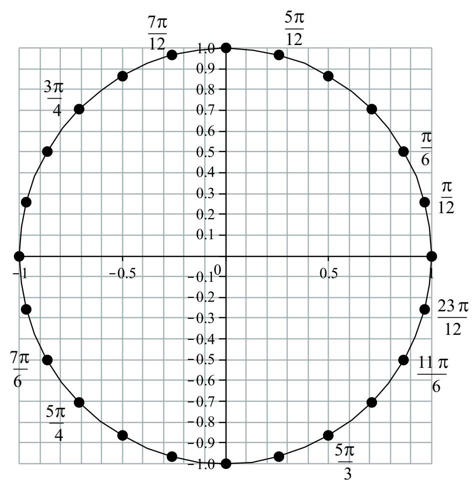


Figure 1.4: Points on the unit circle

Using Figure 1.4, approximate the x-coordinate and the y-coordinate of each of the following:

1. The point on the unit circle that corresponds to $t = \dfrac{\pi}{3}$.

2. The point on the unit circle that corresponds to $t = \dfrac{2\pi}{3}$.

3. The point on the unit circle that corresponds to $t = \dfrac{4\pi}{3}$.

4. The point on the unit circle that corresponds to $t = \dfrac{5\pi}{3}$.

5. The point on the unit circle that corresponds to $t = \dfrac{\pi}{4}$.

6. The point on the unit circle that corresponds to $t = \dfrac{7\pi}{4}$.

Arcs on the Unit Circle

When we wrap the number line around the unit circle, any closed interval on the number line gets mapped to a continuous piece of the unit circle. These pieces are called **arcs** of the circle. For example, the segment $\left[0, \dfrac{\pi}{2}\right]$ on the number line gets mapped to the arc connecting the points $(1, 0)$ and $(0, 1)$ on the unit circle as shown in Figure 1.5. In general, when a closed interval $[a, b]$ is mapped to an arc on the unit circle, the point corresponding to $t = a$ is called the **initial point of the arc**, and the point corresponding to $t = b$ is called the **terminal point of the arc**. So the arc corresponding to the closed interval $\left[0, \dfrac{\pi}{2}\right]$ has initial point $(1, 0)$ and terminal point $(0, 1)$.

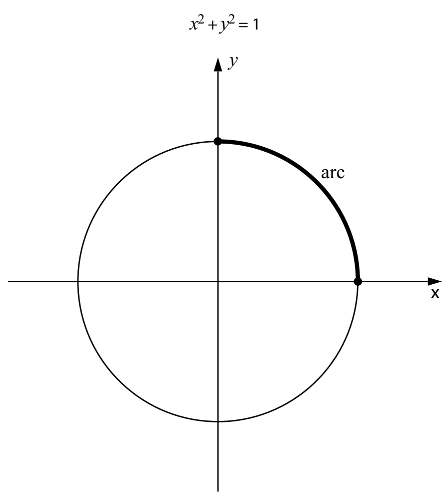


Figure 1.5: An arc on the unit circle

Progress Check 1.3 (Arcs on the Unit Circle).
Draw the following arcs on the unit circle.

1. The arc that is determined by the interval $\left[0, \dfrac{\pi}{4}\right]$ on the number line.

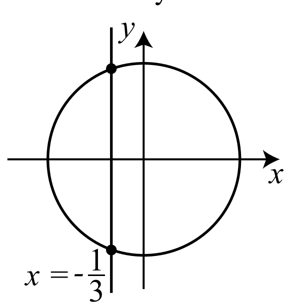
2. The arc that is determined by the interval $\left[0, \dfrac{2\pi}{3}\right]$ on the number line.

3. The arc that is determined by the interval $[0, -\pi]$ on the number line.

Coordinates of Points on the Unit Circle

When we have an equation (usually in terms of x and y) for a curve in the plane and we know one of the coordinates of a point on that curve, we can use the equation to determine the other coordinate for the point on the curve. The equation for the unit circle is $x^2 + y^2 = 1$. So if we know one of the two coordinates of a point on the unit circle, we can substitute that value into the equation and solve for the value(s) of the other variable.

For example, suppose we know that the x-coordinate of a point on the unit circle is $-\dfrac{1}{3}$. This is illustrated on the following diagram. This diagram shows the unit circle ($x^2 + y^2 = 1$) and the vertical line $x = -\dfrac{1}{3}$. This shows that there are two points on the unit circle whose x-coordinate is $-\dfrac{1}{3}$. We can find the y-coordinates by substituting the x-value into the equation and solving for y.



$$
\begin{aligned}
x^2 + y^2 &= 1 \\
\left(-\frac{1}{3}\right)^2 + y^2 &= 1 \\
\frac{1}{9} + y^2 &= 1 \\
y^2 &= \frac{8}{9}
\end{aligned}
$$

Since $y^2 = \dfrac{8}{9}$, we see that $y = \pm\sqrt{\dfrac{8}{9}}$ and so $y = \pm\dfrac{\sqrt{8}}{3}$. So the two points on the unit circle whose x-coordinate is $-\dfrac{1}{3}$ are

$$\left(-\frac{1}{3}, \frac{\sqrt{8}}{3}\right), \quad \text{which is in the second quadrant, and}$$

$$\left(-\frac{1}{3}, -\frac{\sqrt{8}}{3}\right), \quad \text{which is in the third quadrant.}$$

The first point is in the second quadrant and the second point is in the third quadrant. We can now use a calculator to verify that $\dfrac{\sqrt{8}}{3} \approx 0.9428$. This seems consistent with the diagram we used for this problem.

Progress Check 1.4 (Points on the Unit Circle.)

1. Find all points on the unit circle whose y-coordinate is $\dfrac{1}{2}$.

2. Find all points on the unit circle whose x-coordinate is $\dfrac{\sqrt{5}}{4}$.

Summary of Section 1.1

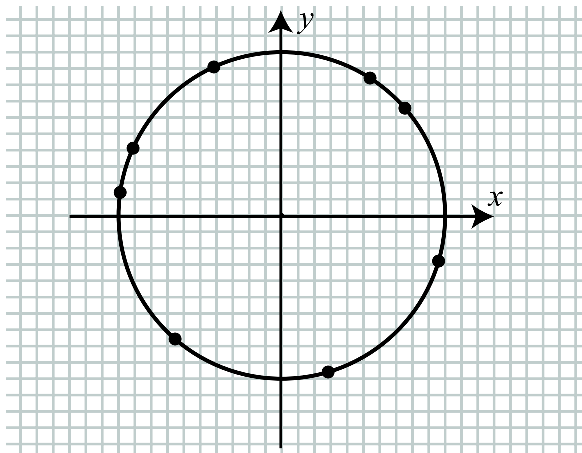
In this section, we studied the following important concepts and ideas:

- The **unit circle** is the circle of radius 1 that is centered at the origin. The equation of the unit circle is $x^2 + y^2 = 1$. It is important because we will use this as a tool to model periodic phenomena.
- We "wrap" the number line about the unit circle by drawing a number line that is tangent to the unit circle at the point $(1, 0)$. We wrap the positive part of the number line around the unit circle in the counterclockwise direction and wrap the negative part of the number line around the unit circle in the clockwise direction.
- When we wrap the number line around the unit circle, any closed interval of real numbers gets mapped to a continuous piece of the unit circle, which is called an arc of the circle. When the closed interval $[a, b]$ is mapped to an arc on the unit circle, the point corresponding to $t = a$ is called **the initial point of the arc**, and the point corresponding to $t = b$ is called **the terminal point of the arc**.

Exercises for Section 1.1

1. The following diagram shows eight points plotted on the unit circle. These points correspond to the following values when the number line is wrapped around the unit circle.

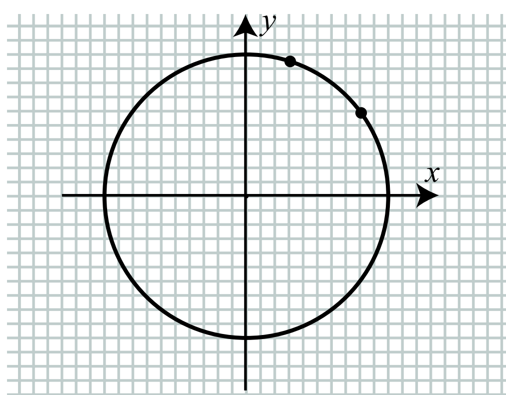
$$t = 1, t = 2, t = 3, t = 4, t = 5, t = 6, t = 7, \text{ and } t = 9.$$



(a) Label each point in the diagram with its value of t.

⋆ **(b)** Approximate the coordinates of the points corresponding to $t = 1$, $t = 5$, and $t = 9$.

⋆ **2.** The following diagram shows the points corresponding to $t = \frac{\pi}{5}$ and $t = \frac{2\pi}{5}$ when the number line is wrapped around the unit circle.



On this unit circle, draw the points corresponding to $t = \frac{4\pi}{5}$, $t = \frac{6\pi}{5}$, $t = \frac{8\pi}{5}$, and $t = \frac{10\pi}{5}$.

3. Draw the following arcs on the unit circle.

(**a**) The arc that is determined by the interval $\left[0, \frac{\pi}{6}\right]$ on the number line.

(**b**) The arc that is determined by the interval $\left[0, \frac{7\pi}{6}\right]$ on the number line.

(**c**) The arc that is determined by the interval $[0, -\frac{\pi}{3}]$ on the number line.

(**d**) The arc that is determined by the interval $[0, -\frac{4\pi}{5}]$ on the number line.

⋆ **4.** Determine the quadrant that contains the terminal point of each given arc with initial point $(1, 0)$ on the unit circle.

(**a**) $\frac{7\pi}{4}$

(**b**) $-\frac{7\pi}{4}$

(**c**) $\frac{3\pi}{5}$

(**d**) $\frac{-3\pi}{5}$

(**e**) $\frac{7\pi}{3}$

(**f**) $\frac{-7\pi}{3}$

(**g**) $\frac{5\pi}{8}$

(**h**) $\frac{-5\pi}{8}$

(**i**) 2.5

(**j**) -2.5

(**k**) 3

(**l**) $3 + 2\pi$

(**m**) $3 - \pi$

(**n**) $3 - 2\pi$

5. Find all the points on the unit circle:

⋆ **(a)** Whose x-coordinate is $\frac{1}{3}$.

⋆ **(b)** Whose y-coordinate is $-\frac{1}{2}$.

(c) Whose x-coordinate is $-\frac{3}{5}$.

(d) Whose y-coordinate is $-\frac{3}{4}$ and whose x-coordinate is negative.

1.2 The Cosine and Sine Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- If the real number t represents the (signed) length of an arc, how do we define $\cos(t)$ and $\sin(t)$?
- In what quadrants (of the terminal point of an arc t on the unit circle) is $\cos(t)$ positive (negative)? In what quadrants (of the terminal point of an arc t on the unit circle) is $\sin(t)$ positive (negative)?
- What is the Pythagorean Identity? How is this identity derived from the equation for the unit circle?

Beginning Activity

1. What is the unit circle? What is the equation of the unit circle?
2. Review Progress Check 1.4 on page 9.
3. Review the completed version of Figure 1.4 that is in the answers for Progress Check 1.2 on page 6.
4. (**a**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, \dfrac{\pi}{2}\right]$?

   (**b**) What is the terminal point of the arc on the unit circle that corresponds to the interval $[0, \pi]$?

   (**c**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, \dfrac{3\pi}{2}\right]$?

   (**d**) What is the terminal point of the arc on the unit circle that corresponds to the interval $\left[0, -\dfrac{\pi}{2}\right]$?

The Cosine and Sine Functions

We started our study of trigonometry by learning about the unit circle, how to wrap the number line around the unit circle, and how to construct arcs on the unit circle. We are now able to use these ideas to define the two major circular, or trigonometric, functions. These circular functions will allow us to model periodic phenomena such as tides, the amount of sunlight during the days of the year, orbits of planets, and many others.

It may seem like the unit circle is a fairly simple object and of little interest, but mathematicians can almost always find something fascinating in even such simple objects. For example, we define the two major circular functions, the *cosine* and *sine*[1] in terms of the unit circle as follows. Figure 1.6 shows an arc of length t on the unit circle. This arc begins at the point $(1, 0)$ and ends at its terminal point $P(t)$. We then define the cosine and sine of the arc t as the x and y coordinates of the point P, so that $P(t) = (\cos(t), \sin(t))$ (we abbreviate the cosine as cos and the sine as sin). So the cosine and sine values are determined by the arc t and the cosine

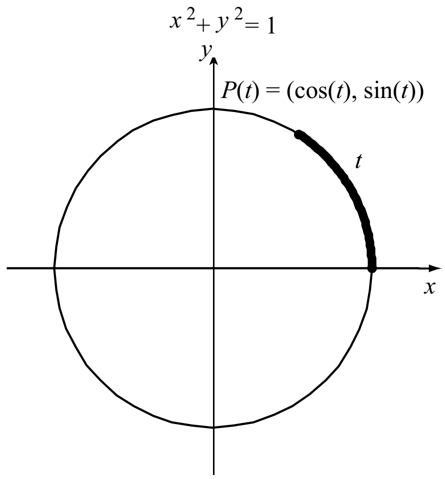


Figure 1.6: The Circular Functions

[1]According to the web site *Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html, the origin of the word sine is Sanskrit through Arabic and Latin. While the accounts of the actual origin differ, it appears that the Sanskrit work "jya" (chord) was taken into Arabic as "jiba", but was then translated into Latin as "jaib" (bay) which became "sinus" (bay or curve). This word was then anglicized to become our "sine". The word cosine began with Plato of Tivoli who use the expression "chorda residui". While the Latin word chorda was a better translation of the Sanskrit-Arabic word for sine than the word sinus, that word was already in use. Thus, "chorda residui" became "cosine".

and sine are *functions* of the arc t. Since the arc lies on the unit circle, we call the cosine and sine ***circular functions***. An important part of trigonometry is the study of the cosine and sine and the periodic phenomena that these functions can model. This is one reason why the circular functions are also called the ***trigonometric functions***.

Note: In mathematics, we always create formal definitions for objects we commonly use. Definitions are critically important because with agreed upon definitions, everyone will have a common understanding of what the terms mean. Without such a common understanding, there would be a great deal of confusion since different people would have different meanings for various terms. So careful and precise definitions are necessary in order to develop mathematical properties of these objects. In order to learn and understand trigonometry, a person needs to be able to explain how the circular functions are defined. So now is a good time to start working on understanding these definitions.

Definition. If the real number t is the directed length of an arc (either positive or negative) measured on the unit circle $x^2 + y^2 = 1$ (with counterclockwise as the positive direction) with initial point (1, 0) and terminal point (x, y), then the **cosine** of t, denoted $\cos(t)$, and **sine** of t, denoted $\sin(t)$, are defined to be

$$\cos(t) = x \qquad \text{and} \qquad \sin(t) = y.$$

Figure 1.6 illustrates these definitions for an arc whose terminal point is in the first quadrant.

At this time, it is not possible to determine the exact values of the cosine and sine functions for specific values of t. It can be done, however, if the terminal point of an arc of length t lies on the x-axis or the y-axis. For example, since the circumference of the unit circle is 2π, an arc of length $t = \pi$ will have it terminal point half-way around the circle from the point $(1, 0)$. That is, the terminal point is at $(-1, 0)$. Therefore,

$$\cos(\pi) = -1 \qquad \text{and} \qquad \sin(\pi) = 0.$$

Progress Check 1.5 (Cosine and Sine Values).
Determine the exact values of each of the following:

1. $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

2. $\cos\left(\frac{3\pi}{2}\right)$ and $\sin\left(\frac{3\pi}{2}\right)$.

3. $\cos(0)$ and $\sin(0)$.

4. $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$.

5. $\cos(2\pi)$ and $\sin(2\pi)$.

6. $\cos(-\pi)$ and $\sin(-\pi)$.

Important Note: Since the cosine and sine are functions of an arc whose length is the real number t, the input t determines the output of the cosine and sine functions. As a result, it is necessary to specify the input value when working with the cosine and sine. In other words, we ALWAYS write $\cos(t)$ where t is the real number input, and NEVER just cos. To reiterate, the cosine and sine are functions, so we MUST indicate the input to these functions.

Progress Check 1.6 (Approximating Cosine and Sine Values).
For this progress check, we will use the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle*. A web address for this applet is

http://gvsu.edu/s/JY

For this applet, we control the value of the input t with the slider for t. The values of t range from -20 to 20 in increments of 0.5. For a given value of t, an arc is drawn of length t and the approximate coordinates of the terminal point of that arc are displayed. Use this applet to find approximate values for each of the following:

1. $\cos(1)$ and $\sin(1)$.

2. $\cos(2)$ and $\sin(2)$.

3. $\cos(-4)$ and $\sin(-4)$.

4. $\cos(5.5)$ and $\sin(5.5)$.

5. $\cos(15)$ and $\sin(15)$.

6. $\cos(-15)$ and $\sin(-15)$.

Some Properties of the Cosine and Sine Functions

The cosine and sine functions are called **circular functions** because their values are determined by the coordinates of points on the unit circle. For each real number t, there is a corresponding arc starting at the point $(1, 0)$ of (directed) length t that lies on the unit circle. The coordinates of the end point of this arc determines the values of $\cos(t)$ and $\sin(t)$.

In previous mathematics courses, we have learned that the domain of a function is the set of all inputs that give a defined output. We have also learned that the range of a function is the set of all possible outputs of the function.

Progress Check 1.7 (Domain and Range of the Circular Functions.)

1. What is the domain of the cosine function? Why?

2. What is the domain of the sine function? Why?

3. What is the largest x coordinate that a point on the unit circle can have? What is the smallest x coordinate that a point on the unit circle can have? What does this tell us about the range of the cosine function? Why?

4. What is the largest y coordinate that a point on the unit circle can have? What is the smallest y coordinate that a point on the unit circle can have? What does this tell us about the range of the sine function? Why?

Although we may not be able to calculate the exact values for many inputs for the cosine and sine functions, we can use our knowledge of the coordinate system and its quadrants to determine if certain values of cosine and sine are positive or negative. The idea is that the signs of the coordinates of a point $P(x, y)$ that is plotted in the coordinate plane are determined by the quadrant in which the point lies. (Unless it lies on one of the axes.) Figure 1.7 summarizes these results for the signs of the cosine and sine function values. The left column in the table is for the location of the terminal point of an arc determined by the real number t.

Quadrant	$\cos(t)$	$\sin(t)$
QI	positive	positive
QII	negative	positive
QIII	negative	negative
QIV	positive	negative

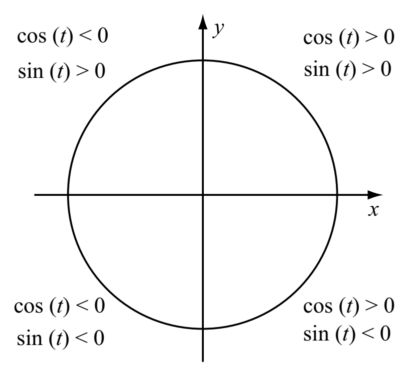


Figure 1.7: Signs of the cosine and sine functions

What we need to do now is to determine in which quadrant the terminal point of an arc determined by a real number t lies. We can do this by once again using the fact that the circumference of the unit circle is 2π, and when we move around the unit circle from the point $(1, 0)$ in the positive (counterclockwise) direction, we will intersect one of the coordinate axes every quarter revolution. For example, if $0 < t < \dfrac{\pi}{2}$, the terminal point of the arc determined by t is in the first quadrant and $\cos(t) > 0$ and $\sin(t) > 0$.

Progress Check 1.8 (Signs of $\cos(t)$ and $\sin(t)$.)

1. If $\dfrac{\pi}{2} < t < \pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
2. If $\pi < t < \dfrac{3\pi}{2}$, then what are the signs of $\cos(t)$ and $\sin(t)$?
3. If $\dfrac{3\pi}{2} < t < 2\pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
4. If $\dfrac{5\pi}{2} < t < 3\pi$, then what are the signs of $\cos(t)$ and $\sin(t)$?
5. For which values of t (between 0 and 2π) is $\cos(t)$ positive? Why?
6. For which values of t (between 0 and 2π) is $\sin(t)$ positive? Why?
7. For which values of t (between 0 and 2π) is $\cos(t)$ negative? Why?
8. For which values of t (between 0 and 2π) is $\sin(t)$ negative? Why?

Progress Check 1.9 (Signs of $\cos(t)$ and $\sin(t)$ (Part 2))
Use the results summarized in Figure 1.7 to help determine if the following quantities are positive, negative, or zero. (Do not use a calculator.)

1. $\cos\left(\dfrac{\pi}{5}\right)$
2. $\sin\left(\dfrac{\pi}{5}\right)$
3. $\cos\left(\dfrac{5\pi}{8}\right)$
4. $\sin\left(\dfrac{5\pi}{8}\right)$
5. $\cos\left(\dfrac{-9\pi}{16}\right)$
6. $\sin\left(\dfrac{-9\pi}{16}\right)$
7. $\cos\left(\dfrac{-25\pi}{12}\right)$
8. $\sin\left(\dfrac{-25\pi}{12}\right)$

The Pythagorean Identity

In mathematics, an **identity** is a statement that is true for all values of the variables for which it is defined. In previous courses, we have worked with algebraic identities such as

$$7x + 12x = 19x \qquad\qquad a + b = b + a$$
$$a^2 - b^2 = (a + b)(a - b) \qquad\qquad x(y + z) = xy + xz$$

where it is understood that all the variables represent real numbers. In trigonometry, we will develop many so-called trigonometric identities. The following progress check introduces one such identity between the cosine and sine functions.

Progress Check 1.10 (Introduction to the Pythagorean Identity)
We know that the equation for the unit circle is $x^2 + y^2 = 1$. We also know that if t is an real number, then the terminal point of the arc determined by t is the point $(\cos(t), \sin(t))$ and that this point lies on the unit circle. Use this information to develop an identity involving $\cos(t)$ and $\sin(t)$.

Using the definitions $x = \cos(t)$ and $y = \sin(t)$ along with the equation for the unit circle, we obtain the following identity, which is perhaps the most important trigonometric identity.

$$\text{For each real number } t, ((\cos(t))^2 + (\sin(t))^2 = 1.$$

This is called the **Pythagorean Identity**. We often use the shorthand notation $\cos^2(t)$ for $(\cos(t))^2$ and $\sin^2(t)$ for $(\sin(t))^2$ and write

$$\text{For each real number } t, \cos^2(t) + \sin^2(t) = 1.$$

Important Note about Notation. Always remember that by $\cos^2(t)$ we mean $(\cos(t))^2$. In addition, note that $\cos^2(t)$ is different from $\cos(t^2)$.

The Pythagorean Identity allows us to determine the value of $\cos(t)$ or $\sin(t)$ if we know the value of the other one and the quadrant in which the terminal point of arc t lies. This is illustrated in the next example.

Example 1.11 (Using the Pythagorean Identity)
Assume that $\cos(t) = \frac{2}{5}$ and the terminal point of arc (t) lies in the fourth quadrant. We will use this information to determine the value of $\sin(t)$. The primary tool we will use is the Pythagorean Identity, but please keep in mind that the terminal point

for the arc t is the point $(\cos(t), \sin(t))$. That is, $x = \cos(t)$ and $y = \sin(t)$. So this problem is very similar to using the equation $x^2 + y^2 = 1$ for the unit circle and substituting $x = \dfrac{2}{5}$.

Using the Pythagorean Identity, we then see that

$$\begin{aligned}
\cos^2(t) + \sin^2(t) &= 1 \\
\left(\frac{2}{5}\right)^2 + \sin^2(t) &= 1 \\
\frac{4}{25} + \sin^2(t) &= 1 \\
\sin^2(t) &= 1 - \frac{4}{25} \\
\sin^2(t) &= \frac{21}{25}
\end{aligned}$$

This means that $\sin(t) = \pm\sqrt{\dfrac{21}{25}}$, and since the terminal point of arc (t) is in the fourth quadrant, we know that $\sin(t) < 0$. Therefore, $\sin(t) = -\sqrt{\dfrac{21}{25}}$. Since $\sqrt{25} = 5$, we can write

$$\sin(t) = -\frac{\sqrt{21}}{\sqrt{25}} = -\frac{\sqrt{21}}{5}.$$

Progress Check 1.12 (Using the Pythagorean Identity)

1. If $\cos(t) = \dfrac{1}{2}$ and the terminal point of the arc t is in the fourth quadrant, determine the value of $\sin(t)$.

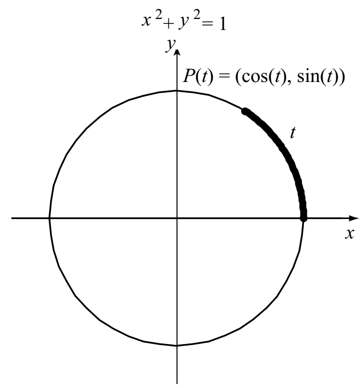
2. If $\sin(t) = -\dfrac{2}{3}$ and $\pi < t < \dfrac{3\pi}{2}$, determine the value of $\cos(t)$.

Summary of Section 1.2

In this section, we studied the following important concepts and ideas:

- If the real number t is the directed length of an arc (either positive or negative) measured on the unit circle $x^2 + y^2 = 1$ (with counterclockwise as the positive direction) with initial point (1, 0) and terminal point (x, y), then

$$\cos(t) = x \quad \text{and} \quad \sin(t) = y.$$



- The signs of $\cos(t)$ and $\sin(t)$ are determined by the quadrant in which the terminal point of an arc t lies.

Quadrant	$\cos(t)$	$\sin(t)$
QI	positive	positive
QII	negative	positive
QIII	negative	negative
QIV	positive	negative

- One of the most important identities in trigonometry, called **the Pythagorean Identity**, is derived from the equation for the unit circle and states:

$$\text{For each real number } t, \cos^2(t) + \sin^2(t) = 1.$$

Exercises for Section 1.2

* **1.** Fill in the blanks for each of the following:

(**a**) For a real number t, the value of $\cos(t)$ is defined to be the ______-coordinate of the ____________ point of an arc t whose initial point is ____________ on the ____________ whose equation is $x^2 + y^2 = 1$.

(**b**) The domain of the cosine function is ____________.

(**c**) The maximum value of $\cos(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \leq t < 2\pi$. The minimum value of $\cos(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \leq t < 2\pi$.

(**d**) The range of the cosine function is ______________.

2. (**a**) For a real number t, the value of $\sin(t)$ is defined to be the _______-coordinate of the ______________ point of an arc t whose initial point is ______________ on the ______________ whose equation is $x^2 + y^2 = 1$.

(**b**) The domain of the sine function is ______________.

(**c**) The maximum value of $\sin(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \leq t < 2\pi$. The minimum value of $\sin(t)$ is ______________ and this occurs at $t =$ ______________ for $0 \leq t < 2\pi$.

(**d**) The range of the sine function is ______________.

3. (**a**) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
0	$(1, 0)$	1	0
$\dfrac{\pi}{2}$			
π			
$\dfrac{3\pi}{2}$			
2π			

(**b**) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
0	$(1, 0)$	1	0
$-\dfrac{\pi}{2}$			
$-\pi$			
$-\dfrac{3\pi}{2}$			
-2π			

(c) Complete the following table of values.

Length of arc on the unit circle	Terminal point of the arc	$\cos(t)$	$\sin(t)$
2π	$(1, 0)$	1	0
$\dfrac{5\pi}{2}$			
3π			
$\dfrac{7\pi}{2}$			
4π			

4. ⋆ **(a)** What are the possible values of $\cos(t)$ if it is known that $\sin(t) = \dfrac{3}{5}$?

(b) What are the possible values of $\cos(t)$ if it is known that $\sin(t) = \dfrac{3}{5}$ and the terminal point of t is in the second quadrant?

⋆ **(c)** What is the value of $\sin(t)$ if it is known that $\cos(t) = -\dfrac{2}{3}$ and the terminal point of t is in the third quadrant?

⋆ **5.** Suppose it is known that $0 < \cos(t) < \dfrac{1}{3}$.

(a) By squaring the expressions in the given inequalities, what conclusions can be made about $\cos^2(t)$?

(b) Use part (a) to write inequalities involving $-\cos^2(t)$ and then inequalities involving $1 - \cos^2(t)$.

(c) Using the Pythagorean identity, we see that $\sin^2(t) = 1 - \cos^2(t)$. Write the last inequality in part (b) in terms of $\sin^2(t)$.

(d) If we also know that $\sin(t) > 0$, what can we now conclude about the value of $\sin(t)$?

6. Use a process similar to the one in exercise (5) to complete each of the following:

(a) Suppose it is known that $-\dfrac{1}{4} < \sin(t) < 0$ and that $\cos(t) > 0$. What can be concluded about $\cos(t)$?

(b) Suppose it is known that $0 \leq \sin(t) \leq \dfrac{3}{7}$ and that $\cos(t) < 0$. What can be concluded about $\cos(t)$?

7. Using the four digit approximations for the cosine and sine values in Progress Check 1.6, calculate each of the following:

- $\cos^2(1) + \sin^2(1)$.
- $\cos^2(2) + \sin^2(2)$.
- $\cos^2(-4) + \sin^2(-4)$.
- $\cos^2(15) + \sin^2(15)$.

What should be the exact value of each of these computations? Why are the results not equal to this exact value?

1.3 Arcs, Angles, and Calculators

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we measure angles using degrees?
- What do we mean by the radian measure of an angle? How is the radian measure of an angle related to the length of an arc on the unit circle?
- Why is radian measure important?
- How do we convert from radians to degrees and from degrees to radians?
- How do we use a calculator to approximate values of the cosine and sine functions?

Introduction

The ancient civilization known as Babylonia was a cultural region based in southern Mesopotamia, which is present-day Iraq. Babylonia emerged as an independent state around 1894 BCE. The Babylonians developed a system of mathematics that was based on a sexigesimal (base 60) number system. This was the origin of the modern day usage of 60 minutes in an hour, 60 seconds in a minute, and 360 degrees in a circle.

Many historians now believe that for the ancient Babylonians, the year consisted of 360 days, which is not a bad approximation given the crudeness of their ancient astronomical tools. As a consequence, they divided the circle into 360 equal length arcs, which gave them a unit angle that was 1/360 of a circle or what we now know as a degree. Even though there are 365.24 days in a year, the Babylonian unit angle is still used as the basis for measuring angles in a circle. Figure 1.8 shows a circle divided up into 6 angles of 60 degrees each, which is also something that fit nicely with the Babylonian base-60 number system.

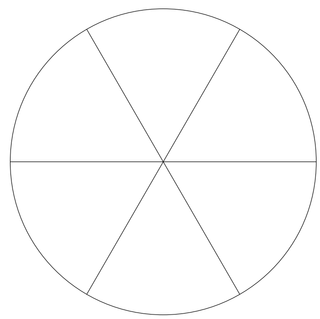


Figure 1.8: A circle with six 60-degree angles.

Angles

We often denote a line that is drawn through 2 points A and B by $\overleftrightarrow{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at the point A and continues indefinitely in the direction of the point B is called **ray *AB*** and is denoted by $\overrightarrow{AB}$. The point A is the **initial point** of the ray $\overrightarrow{AB}$. An **angle** is formed by rotating a ray about its initial point. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the angle. The initial point of the ray is called the **vertex of the angle**.

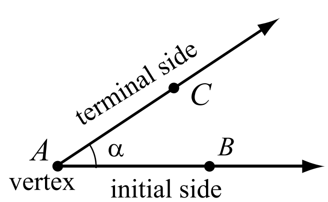


Figure 1.9: An angle including some notation.

Figure 1.9 shows the ray $\overrightarrow{AB}$ rotated about the point A to form an angle. The terminal side of the angle is the ray $\overrightarrow{AC}$. We often refer to this as angle BAC, which is abbreviated as $\angle BAC$. We can also refer to this angle as angle CAB or $\angle CAB$. If we want to use a single letter for this angle, we often use a Greek letter such as α (alpha). We then just say the angle α. Other Greek letters that are often used are β (beta), γ (*gamma*), θ (theta), ϕ (phi), and ρ (rho).

Arcs and Angles

To define the trigonometric functions in terms of angles, we will make a simple connection between angles and arcs by using the so-called standard position of an angle. When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we say that the angle is in **standard position**. The terminal side of the angle is then in one of the four quadrants or lies along one of the axes. When the terminal side is in one of the four quadrants, the terminal side determines the so-called quadrant designation of the angle. See Figure 1.10.

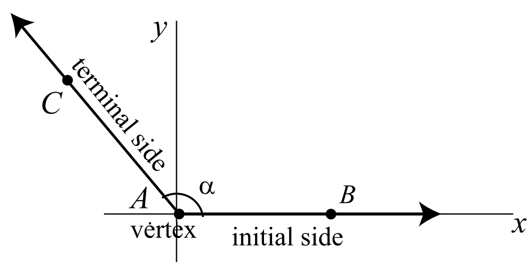


Figure 1.10: Standard position of an angle in the second quadrant.

Progress Check 1.13 (Angles in Standard Position)
Draw an angle in standard position in (1) the first quadrant; (2) the third quadrant; and (3) the fourth quadrant.

If an angle is in standard position, then the point where the terminal side of the angle intersects the unit circle marks the terminal point of an arc as shown in Figure 1.11. Similarly, the terminal point of an arc on the unit circle determines a ray through the origin and that point, which in turn defines an angle in standard position. In this case we say that the angle is *subtended* by the arc. So there is a natural correspondence between arcs on the unit circle and angles in standard position. Because of this correspondence, we can also define the trigonometric functions in terms of angles as well as arcs. Before we do this, however, we need to discuss two different ways to measure angles.

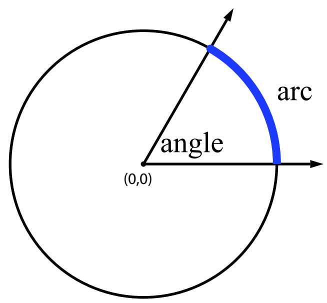


Figure 1.11: An arc and its corresponding angle.

Degrees Versus Radians

There are two ways we will measure angles – in degrees and radians. When we measure the length of an arc, the measurement has a dimension (the length, be it inches, centimeters, or something else). As mentioned in the introduction, the Babylonians divided the circle into 360 regions. So one complete wrap around a circle is 360 degrees, denoted 360°. The unit measure of 1° is an angle that is $1/360$ of the central angle of a circle. Figure 1.8 shows 6 angles of 60° each. The degree ° is a dimension, just like a length. So to compare an angle measured in degrees to an arc measured with some kind of length, we need to connect the dimensions. We can do that with the radian measure of an angle.

Radians will be useful in that a radian is a dimensionless measurement. We want to connect angle measurements to arc measurements, and to do so we will directly define an angle of 1 radian to be an angle subtended by an arc of length 1 (the length of the radius) on the unit circle as shown in Figure 1.12.

> **Definition**. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).

This directly connects angles measured in radians to arcs in that we associate a real number with both the arc and the angle. So an angle of 2 radians cuts off an arc of length 2 on the unit circle, an angle of 3 radians cuts of an arc of length 3 on the unit circle, and so on. Figure 1.13 shows the terminal sides of angles with measures of 0 radians, 1 radian, 2 radians, 3 radians, 4 radians, 5 radians, and 6 radians. Notice that $2\pi \approx 6.2832$ and $6 < 2\pi$ as shown in Figure 1.13.

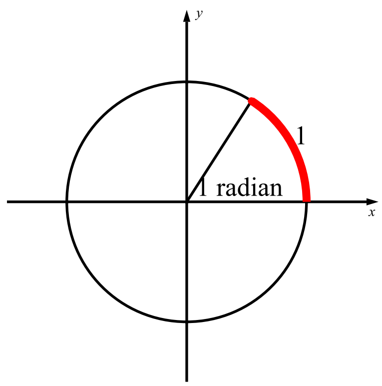


Figure 1.12: One radian.

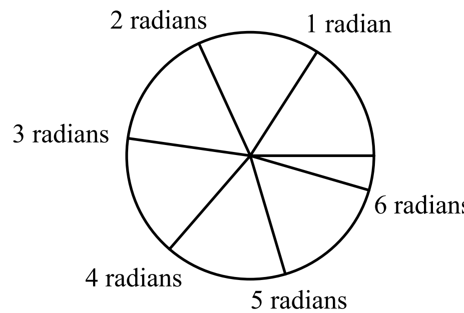


Figure 1.13: Angles with Radian Measure 1, 2, 3, 4, 5, and 6

We can also have angles whose radian measure is negative just like we have arcs with a negative length. The idea is simply to measure in the negative (clockwise) direction around the unit circle. So an angle whose measure is -1 radian is the angle in standard position on the unit circle that is subtended by an arc of length 1 in the negative (clockwise) direction.

So in general, an angle (in standard position) of t radians will correspond to an arc of length t on the unit circle. This allows us to discuss the sine and cosine of an angle measured in radians. That is, when we think of $\sin(t)$ and $\cos(t)$, we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

When we draw a picture of an angle in standard position, we often draw a small arc near the vertex from the initial side to the terminal side as shown in Figure 1.14, which shows an angle whose measure is $\dfrac{3}{4}\pi$ radians.

Progress Check 1.14 (Radian Measure of Angles)

1. Draw an angle in standard position with a radian measure of:

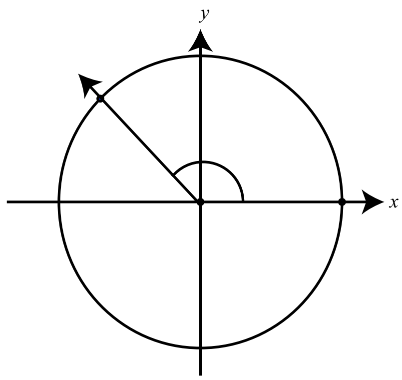


Figure 1.14: An angle with measure $\frac{3}{4}\pi$ in standard position

(**a**) $\dfrac{\pi}{2}$ radians.

(**b**) π radians.

(**c**) $\dfrac{3\pi}{2}$ radians.

(**d**) $-\dfrac{3\pi}{2}$ radians.

2. What is the degree measure of each of the angles in part (1)?

Conversion Between Radians and Degrees

Radian measure is the preferred measure of angles in mathematics for many reasons, the main one being that a radian has no dimensions. However, to effectively use radians, we will want to be able to convert angle measurements between radians and degrees.

Recall that one wrap of the unit circle corresponds to an arc of length 2π, and an arc of length 2π on the unit circle corresponds to an angle of 2π radians. An angle of $360°$ is also an angle that wraps once around the unit circle, so an angle of $360°$ is equivalent to an angle of 2π radians, or

- each degree is $\dfrac{\pi}{180}$ radians,
- each radian is $\dfrac{180}{\pi}$ degrees.

Notice that 1 radian is then $\dfrac{180}{\pi} \approx 57.3°$, so a radian is quite large compared to a degree. These relationships allow us to quickly convert between degrees and radians. For example:

- If an angle has a degree measure of 35 degrees, then its radian measure can be calculated as follows:

  $$35 \text{ degrees} \times \frac{\pi \text{ radians}}{180 \text{ degrees}} = \frac{35\pi}{180}\text{radians.}$$

  Rewriting this fraction, we see that an angle with a measure of 35 degrees has a radian measure of $\dfrac{7\pi}{36}$ radians.

- If an angle has a radian measure of $\dfrac{3\pi}{10}$ radians, then its degree measure can be calculated as follows:

  $$\frac{3\pi}{10}\text{radians} \times \frac{180 \text{ degrees}}{\pi \text{ radians}} = \frac{540}{10}\text{degrees.}$$

  So an angle with a radian measure of $\dfrac{3\pi}{10}$ has an angle measure of 54°.

IMPORTANT NOTE: Since a degree is a dimension, we MUST include the degree mark ° whenever we write the degree measure of an angle. A radian has no dimension so there is no dimension mark to go along with it. Consequently, if we write 2 for the measure of an angle we understand that the angle is measured in radians. If we really mean an angle of 2 degrees, then we must write 2°.

Progress Check 1.15 (Radian-Degree Conversions)
Complete Table 1.1 converting from degrees to radians and vice versa.

Calculators and the Trigonometric Functions

We have now seen that when we think of $\sin(t)$ or $\cos(t)$, we can think of t as a real number, the length of an arc, or the radian measure of an angle. In Section 1.5, we will see how to determine the exact values of the cosine and sine functions for a few special arcs (or angles). For example, we will see that $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$. However, the definition of cosine and sine as coordinates of points on the unit circle makes it difficult to find exact values for these functions except at very special arcs. While exact values are always best, technology plays an important role in allowing us to

Angle in radians	Angle in degrees
0	$0°$
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	$90°$
	$120°$
$\frac{3\pi}{4}$	$135°$
	$150°$
	$180°$

Angle in radians	Angle in degrees
$\frac{7\pi}{6}$	
$\frac{5\pi}{4}$	
$\frac{4\pi}{3}$	
$\frac{3\pi}{2}$	$270°$
	$300°$
	$315°$
	$330°$
2π	$360°$

Table 1.1: Conversions between radians and degrees.

approximate the values of the circular (or trigonometric)functions. Most hand-held calculators, calculators in phone or tablet apps, and online calculators have a cosine key and a sine key that you can use to approximate values of these functions, but we must keep in mind that the calculator only provides an *approximation* of the value, not the exact value (except for a small collection of arcs). In addition, most calculators will approximate the sine and cosine of angles.

To do this, the calculator has two modes for angles: Radian and Degree. Because of the correspondence between real numbers, length of arcs, and radian measures of angles, for now, we will always put our calculators in radian mode. In fact, we have seen that an angle measured in radians subtends an arc of that radian measure along the unit circle. **So the cosine or sine of an angle measured in radians is the same thing as the cosine or sine of a real number when that real number is interpreted as the length of an arc along the unit circle**. (When we study the trigonometry of triangles in Chapter 3, we will use the degree mode. For an introductory discussion of the trigonometric functions of an angle measure in degrees, see Exercise (4)).

Progress Check 1.16 (Using a Calculator)
In Progress Check 1.6, we used the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* at `http://gvsu.edu/s/JY` to approximate the values of the cosine and sine functions at certain values. For example, we found that

- $\cos(1) \approx 0.5403$,
  $\sin(1) \approx 0.8415$.
- $\cos(2) \approx -0.4161$
  $\sin(2) \approx 0.9093$.
- $\cos(-4) \approx -0.6536$
  $\sin(-4) \approx 0.7568$.
- $\cos(-15) \approx -0.7597$
  $\sin(-15) \approx -0.6503$.

Use a calculator to determine these values of the cosine and sine functions and compare the values to the ones above. Are they the same? How are they different?

Summary of Section 1.3

In this section, we studied the following important concepts and ideas:

- An **angle** is formed by rotating a ray about its initial point. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The initial point of the ray is called the **vertex of the angle**.

- When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**.

- There are two ways to measure angles. For degree measure, one complete wrap around a circle is 360 degrees, denoted 360°. The unit measure of 1° is an angle that is $1/360$ of the central angle of a circle. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).

- We convert the measure of an angle from degrees to radians by using the fact that each degree is $\dfrac{\pi}{180}$ radians. We convert the measure of an angle from radians to degrees by using the fact that each radian is $\dfrac{180}{\pi}$ degrees.

Exercises for Section 1.3

1. Convert each of the following degree measurements for angles into radian measures for the angles. In each case, first write the result as a fractional multiple of π and then use a calculator to obtain a 4 decimal place approximation of the radian measure.

⋆ **(a)** $15°$
⋆ **(b)** $58°$
(c) $112°$
(d) $210°$
⋆ **(e)** $-40°$
(f) $-78°$

2. Convert each of the following radian measurements for angles into degree measures for the angles. When necessary, write each result as a 4 decimal place approximation.

⋆ **(a)** $\frac{3}{8}\pi$ radians
⋆ **(b)** $\frac{9}{7}\pi$ radians
(c) $-\frac{7}{15}\pi$ radians
⋆ **(d)** 1 radian
(e) 2.4 radians
(f) 3 radians

3. Draw an angle in standard position of an angle whose radian measure is:

(a) $\frac{1}{4}\pi$
(b) $\frac{1}{3}\pi$
(c) $\frac{2}{3}\pi$
(d) $\frac{5}{4}\pi$
(e) $-\frac{1}{3}\pi$
(f) 3.4

4. In Progress Check 1.16, we used the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* to approximate values of the cosine and sine functions. We will now do something similar to approximate the cosine and sine values for angles measured in degrees.

We have seen that the terminal side of an angle in standard position intersects the unit circle in a point. We use the coordinates of this point to determine the cosine and sine of that angle. When the angle is measured in radians, the radian measure of the angle is the same as the arc on the unit circle subtended by the angle. This is not true when the angle is measure in degrees, but we can still use the intersection point to define the cosine and sine of the angle. So if an angle in standard position has degree measurement $a°$, then we define $\cos(a°)$ to be the x-coordinate of the point of intersection of the terminal side of that angle and the unit circle. We define $\sin(a°)$ to be the

y-coordinate of the point of intersection of the terminal side of that angle and the unit circle.

We will now use the Geogebra applet *Angles and the Unit Circle*. A web address for this applet is

http://gvsu.edu/s/VG

For this applet, we control the value of the input $a°$ with the slider for a. The values of a range from $-180°$ to $180°$ in increments of $5°$. For a given value of $a°$, an angle in standard position is drawn and the coordinates of the point of intersection of the terminal side of that angle and the unit circle are displayed. Use this applet to approximate values for each of the following:

⋆ **(a)** $\cos(10°)$ and $\sin(10°)$.

(b) $\cos(60°)$ and $\sin(60°)$.

(c) $\cos(135°)$ and $\sin(135°)$.

⋆ **(d)** $\cos(-10°)$ and $\sin(-10°)$.

(e) $\cos(-135°)$ and $\sin(-135°)$.

(f) $\cos(85°)$ and $\sin(85°)$.

5. Exercise (4) must be completed before doing this exercise. Put the calculator you are using in Degree mode. Then use the calculator to determine approximate values of the cosine and sine functions in Exercise (4). Are the values the same? How are they different?

1.4 Velocity and Angular Velocity

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is arc length?
- What is the difference between linear velocity and angular velocity?
- What are the formulas that relate linear velocity to angular velocity?

Beginning Activity

1. What is the formula for the circumference C of a circle whose radius is r?

2. Suppose person A walks along the circumference of a circle with a radius of 10 feet, and person B walks along the circumference of a circle of radius 20 feet. Also, suppose it takes both A and B 1 minute to walk one-quarter of the circumference of their respective circles (one-quarter of a complete revolution). Who walked the most distance in one minute?

3. Suppose both people continue walking at the same pace they did for the first minute. How many complete revolutions of the circle will each person walk in 8 minutes? In 10 minutes?

Arc Length on a Circle

In Section 1.3, we learned that the radian measure of an angle was equal to the length of the arc on the unit circle associated with that angle. So an arc of length 1 on the unit circle subtends an angle of 1 radian. There will be times when it will also be useful to know the length of arcs on other circles that subtend the same angle.

In Figure 1.15, the inner circle has a radius of 1, the outer circle has a radius of r, and the angle shown has a measure of θ radians. So the arc length on the unit

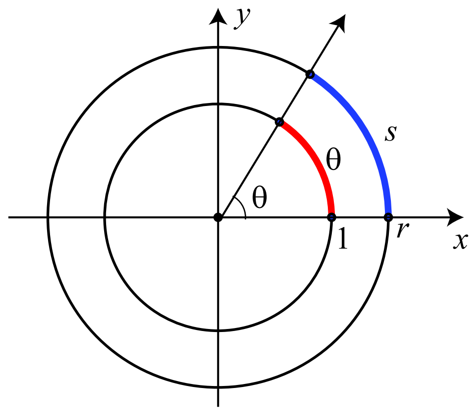


Figure 1.15: Arcs subtended by an angle of 1 radian.

circle subtended by the angle is θ, and we have used s to represent the arc length on the circle of radius r subtended by the angle.

Recall that the circumference of a circle of radius r is $2\pi r$ while the circumference of the circle of radius 1 is 2π. Therefore, the ratio of an arc length s on the circle of radius r that subtends an angle of θ radians to the corresponding arc on the unit circle is $\dfrac{2\pi r}{2\pi} = r$. So it follows that

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi} = r$$
$$s = r\theta$$

Definition. On a circle of radius r, the **arc length** s intercepted by a central angle with radian measure θ is

$$s = r\theta.$$

Note: It is important to remember that to calculate arc length[2] , we must measure the central angle in radians.

[2]It is not clear why the letter s is usually used to represent arc length. One explanation is that the arc "subtends" an angle.

Progress Check 1.17 (Using the Formula for Arc Length)
Using the circles in the beginning activity for this section:

1. Use the formula for arc length to determine the arc length on a circle of radius 10 feet that subtends a central angle of $\dfrac{\pi}{2}$ radians. Is the result equal to one-quarter of the circumference of the circle?
2. Use the formula for arc length to determine the arc length on a circle of radius 20 feet that subtends a central angle of $\dfrac{\pi}{2}$ radians. Is the result equal to one-quarter of the circumference of the circle?
3. Determine the arc length on a circle of radius 3 feet that is subtended by an angle of 22°.

Why Radians?

Degree measure is familiar and convenient, so why do we introduce the concept of radian measure for angles? This is a good question, but one with a subtle answer. As we just saw, the length s of an arc on a circle of radius r subtended by angle of θ radians is given by $s = r\theta$, so $\theta = \dfrac{s}{r}$. As a result, a radian measure is a ratio of two lengths (the quotient of the length of an arc by a radius of a circle), which makes radian measure a dimensionless quantity. Thus, a measurement in radians can just be thought of as a real number. This is convenient for dealing with arc length (and angular velocity as we will soon see), and it will also be useful when we study periodic phenomena in Chapter 2. For this reason radian measure is universally used in mathematics, physics, and engineering as opposed to degrees, because when we use degree measure we always have to take the degree dimension into account in computations. This means that radian measure is actually more natural from a mathematical standpoint than degree measure.

Linear and Angular Velocity

The connection between an arc on a circle and the angle it subtends measured in radians allows us to define quantities related to motion on a circle. Objects traveling along circular paths exhibit two types of velocity: *linear* and *angular* velocity. Think of spinning on a merry-go-round. If you drop a pebble off the edge of a moving merry-go-round, the pebble will not drop straight down. Instead, it will continue to move forward with the velocity the merry-go-round had the moment the pebble was released. This is the *linear velocity* of the pebble. The linear velocity measures how the arc length changes over time.

Consider a point P moving at a constant linear velocity along the circumference of a circle of radius r. This is called **uniform circular motion**. Suppose that P moves a distance of s units in time t. The *linear velocity* v of the point P is the distance it traveled divided by the time elapsed. That is, $v = \dfrac{s}{t}$. The distance s is the arc length and we know that $s = r\theta$.

Definition. Consider a point P moving at a constant linear velocity along the circumference of a circle of radius r. The **linear velocity** v of the point P is given by

$$v = \frac{s}{t} = \frac{r\theta}{t},$$

where θ, measured in radians, is the central angle subtended by the arc of length s.

Another way to measure how fast an object is moving at a constant speed on a circular path is called *angular velocity*. Whereas the linear velocity measures how the arc length changes over time, the *angular velocity* is a measure of how fast the central angle is changing over time.

Definition. Consider a point P moving with constant velocity along the circumference of a circle of radius r on an arc that corresponds to a central angle of measure θ (in radians). The **angular velocity** ω of the point is the radian measure of the angle θ divided by the time t it takes to sweep out this angle. That is

$$\omega = \frac{\theta}{t}.$$

Note: The symbol ω is the lower case Greek letter "omega." Also, notice that the angular velocity does not depend on the radius r.

This is a somewhat specialized definition of angular velocity that is slightly different than a common term used to describe how fast a point is revolving around a circle. This term is **revolutions per minute** or **rpm**. Sometimes the unit **revolutions per second** is used. A better way to represent revolutions per minute is to use the "unit fraction" $\dfrac{\text{rev}}{\text{min}}$. Since 1 revolution is 2π radians, we see that if an object is moving at x revolutions per minute, then

$$\omega = x\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\,\text{rad}}{\text{rev}} = x(2\pi)\frac{\text{rad}}{\text{min}}.$$

Progress Check 1.18 (Determining Linear Velocity)
Suppose a circular disk is rotating at a rate of 40 revolutions per minute. We wish to determine the linear velocity v (in feet per second) of a point that is 3 feet from the center of the disk.

1. Determine the angular velocity ω of the point in radians per minute. **Hint**: Use the formula
$$\omega = x\frac{\text{rev}}{\text{min}} \cdot \frac{2\pi\,\text{rad}}{\text{rev}}.$$
2. We now know $\omega = \dfrac{\theta}{t}$. So use the formula $v = \dfrac{r\theta}{t}$ to determine v in feet per minute.
3. Finally, convert the linear velocity v in feet per minute to feet per second.

Notice that in Progress Check 1.18, once we determined the angular velocity, we were able to determine the linear velocity. What we did in this specific case we can do in general. There is a simple formula that directly relates linear velocity to angular velocity. Our formula for linear velocity is $v = \dfrac{s}{t} = \dfrac{r\theta}{t}$. Notice that we can write this is $v = r\dfrac{\theta}{t}$. That is, $v = r\omega$.

> Consider a point P moving with constant (linear) velocity v along the circumference of a circle of radius r. If the angular velocity is ω, then
> $$v = r\omega.$$

So in Progress Check 1.18, once we determined that $\omega = 80\pi\dfrac{\text{rad}}{\text{min}}$, we could determine v as follows:

$$v = r\omega = (3\text{ ft})\left(80\pi\frac{\text{rad}}{\text{min}}\right) = 240\pi\frac{\text{ft}}{\text{min}}.$$

Notice that since radians are “unit-less”, we can drop them when dealing with equations such as the preceding one.

Example 1.19 (Linear and Angular Velocity)

The LP (long play) or $33\frac{1}{3}$ rpm vinyl record is an analog sound storage medium and has been used for a long time to listen to music. An LP is usually 12 inches or 10 inches in diameter. In order to work with our formulas for linear and angular velocity, we need to know the angular velocity in radians per time unit. To do this, we will convert $33\frac{1}{3}$ revolutions per minute to radians per minute. We will use the fact that $33\frac{1}{3} = \frac{100}{3}$.

$$\begin{aligned}\omega &= \frac{100}{3}\frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \\ &= \frac{200\pi}{3}\frac{\text{rad}}{\text{min}}\end{aligned}$$

We can now use the formula $v = r\omega$ to determine the linear velocity of a point on the edge of a 12 inch LP. The radius is 6 inches and so

$$\begin{aligned}v &= r\omega \\ &= (6 \text{ inches})\left(\frac{200\pi}{3}\frac{\text{rad}}{\text{min}}\right) \\ &= 400\pi\frac{\text{inches}}{\text{min}}\end{aligned}$$

It might be more convenient to express this as a decimal value in inches per second. So we get

$$\begin{aligned}v &= 400\pi\frac{\text{inches}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \\ &\approx 20.944\frac{\text{inches}}{\text{sec}}\end{aligned}$$

The linear velocity is approximately 20.944 inches per second.

Progress Check 1.20 (Linear and Angular Velocity)

For these problems, we will assume that the Earth is a sphere with a radius of 3959 miles. As the Earth rotates on its axis, a person standing on the Earth will travel in a circle that is perpendicular to the axis.

1. The Earth rotates on its axis once every 24 hours. Determine the angular velocity of the Earth in radians per hour. (Leave your answer in terms of the number π.)

2. As the Earth rotates, a person standing on the equator will travel in a circle whose radius is 3959 miles. Determine the linear velocity of this person in miles per hour.

3. As the Earth rotates, a person standing at a point whose latitude is 60° north will travel in a circle of radius 2800 miles. Determine the linear velocity of this person in miles per hour and feet per second.

Summary of Section 1.4

In this section, we studied the following important concepts and ideas:

- On a circle of radius r, the **arc length** s intercepted by a central angle with radian measure θ is
$$s = r\theta.$$
- Uniform circular motion is when a point moves at a constant linear velocity along the circumference of a circle. The **linear velocity** is the arc length traveled by the point divided by the time elapsed. Whereas the linear velocity measures how the arc length changes over time, the *angular velocity* is a measure of how fast the central angle is changing over time. The **angular velocity** of the point is the radian measure of the angle divided by the time it takes to sweep out this angle.
- For a point P moving with constant (linear) velocity v along the circumference of a circle of radius r, we have
$$v = r\omega,$$
where ω is the angular velocity of the point.

Exercises for Section 1.4

* **1.** Determine the arc length (to the nearest hundredth of a unit when necessary) for each of the following.

(**a**) An arc on a circle of radius 6 feet that is intercepted by a central angle of $\dfrac{2\pi}{3}$ radians. Compare this to one-third of the circumference of the circle.

(**b**) An arc on a circle of radius 100 miles that is intercepted by a central angle of 2 radians.

(**c**) An arc on a circle of radius 20 meters that is intercepted by a central angle of $\dfrac{13\pi}{10}$ radians.

(**d**) An arc on a circle of radius 10 feet that is intercepted by a central angle of 152 degrees.

2. In each of the following, when it is possible, determine the exact measure of the central angle in radians. Otherwise, round to the nearest hundredth of a radian.

⋆ **(a)** The central angle that intercepts an arc of length 3π feet on a circle of radius 5 feet.

⋆ **(b)** The central angle that intercepts an arc of length 18 feet on a circle of radius 5 feet.

(c) The central angle that intercepts an arc of length 20 meters on a circle of radius 12 meters.

3. In each of the following, when it is possible, determine the exact measure of central the angle in degrees. Otherwise, round to the nearest hundredth of a degree.

⋆ **(a)** The central angle that intercepts an arc of length 3π feet on a circle of radius 5 feet.

⋆ **(b)** The central angle that intercepts an arc of length 18 feet on a circle of radius 5 feet.

(c) The central angle that intercepts an arc of length 20 meters on a circle of radius 12 meters.

(d) The central angle that intercepts an arc of length 5 inches on a circle of radius 5 inches.

(e) The central angle that intercepts an arc of length 12 inches on a circle of radius 5 inches.

4. Determine the distance (in miles) that the planet Mars travels in one week in its path around the sun. For this problem, assume that Mars completes one complete revolution around the sun in 687 days and that the path of Mars around the sun is a circle with a radius of 227.5 million miles.

⋆ **5.** Determine the distance (in miles) that the Earth travels in one day in its path around the sun. For this problem, assume that Earth completes one complete revolution around the sun in 365.25 days and that the path of Earth around the sun is a circle with a radius of 92.96 million miles.

6. A compact disc (CD) has a diameter of 12 centimeters (cm). Suppose that the CD is in a CD-player and is rotating at 225 revolutions per minute. What is the angular velocity of the CD (in radians per second) and what is the linear velocity of a point on the edge of the CD?

7. A person is riding on a Ferris wheel that takes 28 seconds to make a complete revolution. Her seat is 25 feet from the axle of the wheel.

(**a**) What is her angular velocity in revolutions per minute? Radians per minute? Degrees per minute?

(**b**) What is her linear velocity?

(**c**) Which of the quantities angular velocity and linear velocity change if the person's seat was 20 feet from the axle instead of 25 feet? Compute the new value for any value that changes. Explain why each value changes or does not change.

8. A small pulley with a radius of 3 inches is connected by a belt to a larger pulley with a radius of 7.5 inches (See Figure 1.16). The smaller pulley is connected to a motor that causes it to rotate counterclockwise at a rate of 120 rpm (revolutions per minute). Because the two pulleys are connected by the belt, the larger pulley also rotates in the counterclockwise direction.

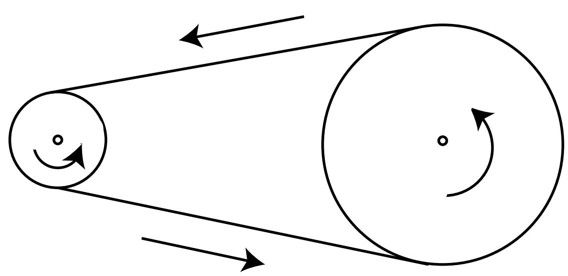


Figure 1.16: Two Pulleys Connected by a Belt

(**a**) Determine the angular velocity of the smaller pulley in radians per minute.

⋆ **(b)** Determine the linear velocity of the rim of the smaller pulley in inches per minute.

(c) What is the linear velocity of the rim of the larger pulley? Explain.

(d) Find the angular velocity of the larger pulley in radians per minute.

(e) How many revolutions per minute is the larger pulley turning?

9. A small pulley with a radius of 10 centimeters inches is connected by a belt to a larger pulley with a radius of 24 centimeters inches (See Figure 1.16). The larger pulley is connected to a motor that causes it to rotate counterclockwise at a rate of 75 rpm (revolutions per minute). Because the two pulleys are connected by the belt, the smaller pulley also rotates in the counterclockwise direction.

(**a**) Determine the angular velocity of the larger pulley in radians per minute.

⋆ **(b)** Determine the linear velocity of the rim of the large pulley in inches per minute.

(c) What is the linear velocity of the rim of the smaller pulley? Explain.

(d) Find the angular velocity of the smaller pulley in radians per second.

(e) How many revolutions per minute is the smaller pulley turning?

10. The radius of a car wheel is 15 inches. If the car is traveling 60 miles per hour, what is the angular velocity of the wheel in radians per minute? How fast is the wheel spinning in revolutions per minute?

11. The mean distance from Earth to the moon is 238,857 miles. Assuming the orbit of the moon about Earth is a circle with a radius of 238,857 miles and that the moon makes one revolution about Earth every 27.3 days, determine the linear velocity of the moon in miles per hour. Research the distance of the moon to Earth and explain why the computations that were just made are approximations.

1.5 Common Arcs and Reference Arcs

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we determine the values for cosine and sine for arcs whose endpoints are on the x-axis or the y-axis?
- What are the exact values of cosine and sine for $t = \dfrac{\pi}{6}$, $t = \dfrac{\pi}{4}$, and $t = \dfrac{\pi}{3}$?
- What is the reference arc for a given arc? How do we determine the reference arc for a given arc?
- How do we use reference arcs to calculate the values of the cosine and sine at other arcs that have $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ as reference arcs?

Beginning Activity

Figure 1.17 shows a unit circle with the terminal points for some arcs between 0 and 2π. In addition, there are four line segments drawn on the diagram that form a rectangle. The line segments go from: (1) the terminal point for $t = \dfrac{\pi}{6}$ to the terminal point for $t = \dfrac{5\pi}{6}$; (2) the terminal point for $t = \dfrac{5\pi}{6}$ to the terminal point for $t = \dfrac{7\pi}{6}$; (3) the terminal point for $t = \dfrac{7\pi}{6}$ to the terminal point for $t = \dfrac{11\pi}{6}$; and (4) the terminal point for $t = \dfrac{11\pi}{6}$ to the terminal point for $t = \dfrac{\pi}{6}$.

1. What are the approximate values of $\cos\left(\dfrac{\pi}{6}\right)$ and $\sin\left(\dfrac{\pi}{6}\right)$?

2. What are the approximate values of $\cos\left(\dfrac{5\pi}{6}\right)$ and $\sin\left(\dfrac{5\pi}{6}\right)$?

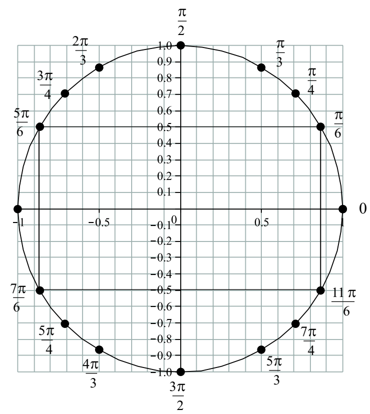


Figure 1.17: Some Arcs on the unit circle

3. What are the approximate values of $\cos\left(\dfrac{7\pi}{6}\right)$ and $\sin\left(\dfrac{7\pi}{6}\right)$?

4. What are the approximate values of $\cos\left(\dfrac{11\pi}{6}\right)$ and $\sin\left(\dfrac{11\pi}{6}\right)$?

5. Draw a similar rectangle on Figure 1.17 connecting the terminal points for $t = \dfrac{\pi}{4}$, $t = \dfrac{3\pi}{4}$, $t = \dfrac{5\pi}{4}$, and $t = \dfrac{7\pi}{4}$. How do the cosine and sine values for these arcs appear to be related?

Our task in this section is to determine the exact cosine and sine values for all of the arcs whose terminal points are shown in Figure 1.17. We first notice that we already know the cosine and sine values for the arcs whose terminal points are on one of the coordinate axes. These values are shown in the following table.

t	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
$\cos(t)$	1	0	-1	0	1
$\sin(t)$	0	1	0	-1	0

Table 1.2: Cosine and Sine Values

The purpose of the beginning activity was to show that we determine the values of cosine and sine for the other arcs by finding only the cosine and sine values for the arcs whose terminal points are in the first quadrant. So this is our first task. To do this, we will rely on some facts about certain right triangles. The three triangles we will use are shown in Figure 1.18.

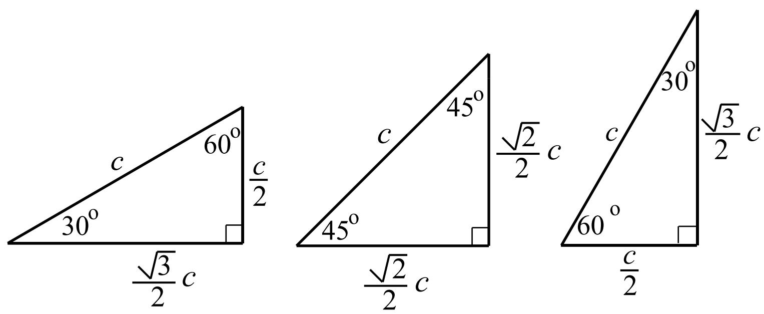


Figure 1.18: Special Right Triangles.

In each figure, the hypotenuse of the right triangle has a length of c units. The lengths of the other sides are determined using the Pythagorean Theorem. An explanation of how these lengths were determined can be found on page 425 in Appendix C. The usual convention is to use degree measure for angles when we work with triangles, but we can easily convert these degree measures to radian measures.

- A $30°$ angle has a radian measure of $\dfrac{\pi}{6}$ radians.
- A $45°$ angle has a radian measure of $\dfrac{\pi}{4}$ radians.
- A $60°$ angle has a radian measure of $\dfrac{\pi}{3}$ radians.

The Values of Cosine and Sine at $t = \dfrac{\pi}{6}$

Figure 1.19 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{6}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{6}\right), \sin\left(\dfrac{\pi}{6}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{6}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{6}\right).$$

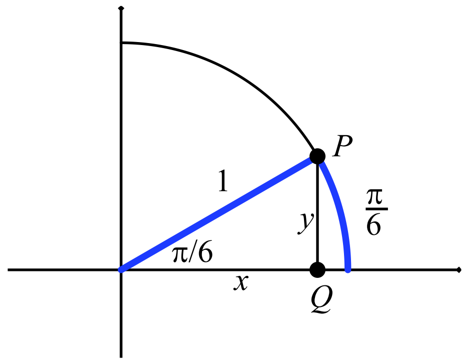


Figure 1.19: The arc $\dfrac{\pi}{6}$ and its associated angle.

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the angle associated with the arc is $\dfrac{\pi}{6}$ radians or 30°. From what we know about this type of right triangle, the other acute angle in the right triangle is 60° or $\dfrac{\pi}{3}$ radians. We can then use the results shown in the triangle on the left in Figure 1.18 to conclude that $x = \dfrac{\sqrt{3}}{2}$ and $y = \dfrac{1}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Progress Check 1.21 (Comparison to the Beginning Activity)
In the beginning activity for this section, we used the unit circle to approximate the values of the cosine and sine functions at $t = \dfrac{\pi}{6}$, $t = \dfrac{5\pi}{6}$, $t = \dfrac{7\pi}{6}$, and $t = \dfrac{11\pi}{6}$. We also saw that these values are all related and that once we have values for the cosine and sine functions at $t = \dfrac{\pi}{6}$, we can use our knowledge of the four quadrants to determine these function values at $t = \dfrac{5\pi}{6}$, $t = \dfrac{7\pi}{6}$, and

$t = \dfrac{11\pi}{6}$. Now that we know that

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2},$$

determine the exact values of each of the following:

1. $\cos\left(\dfrac{5\pi}{6}\right)$ and $\sin\left(\dfrac{5\pi}{6}\right)$.

2. $\cos\left(\dfrac{7\pi}{6}\right)$ and $\sin\left(\dfrac{7\pi}{6}\right)$.

3. $\cos\left(\dfrac{11\pi}{6}\right)$ and $\sin\left(\dfrac{11\pi}{6}\right)$.

The Values of Cosine and Sine at $t = \dfrac{\pi}{4}$

Figure 1.20 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{4}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{4}\right), \sin\left(\dfrac{\pi}{4}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{4}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{4}\right).$$

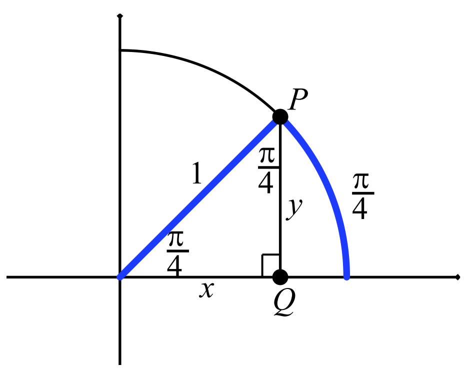


Figure 1.20: The arc $\dfrac{\pi}{4}$ and its associated angle.

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the acute angles are $\dfrac{\pi}{4}$ radians or 45°. We can then use the results shown in the triangle in the middle of Figure 1.18 to conclude that $x = \dfrac{\sqrt{2}}{2}$ and $y = \dfrac{\sqrt{2}}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Progress Check 1.22 (Comparison to the Beginning Activity)
Now that we know that

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

use a method similar to the one used in Progress Check 1.21 to determine the exact values of each of the following:

1. $\cos\left(\dfrac{3\pi}{4}\right)$ and $\sin\left(\dfrac{3\pi}{4}\right)$.
2. $\cos\left(\dfrac{5\pi}{4}\right)$ and $\sin\left(\dfrac{5\pi}{4}\right)$.
3. $\cos\left(\dfrac{7\pi}{4}\right)$ and $\sin\left(\dfrac{7\pi}{4}\right)$.

The Values of Cosine and Sine at $t = \dfrac{\pi}{3}$

Figure 1.21 shows the unit circle in the first quadrant with an arc in standard position of length $\dfrac{\pi}{3}$. The terminal point of the arc is the point P and its coordinates are $\left(\cos\left(\dfrac{\pi}{3}\right), \sin\left(\dfrac{\pi}{3}\right)\right)$. So from the diagram, we see that

$$x = \cos\left(\frac{\pi}{3}\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{3}\right).$$

As shown in the diagram, we form a right triangle by drawing a line from P that is perpendicular to the x-axis and intersects the x-axis at Q. So in this right triangle, the angle associated with the arc is $\dfrac{\pi}{3}$ radians or 60°. From what we know about this type of right triangle, the other acute angle in the right triangle is 30° or $\dfrac{\pi}{6}$ radians. We can then use the results shown in the triangle on the right in

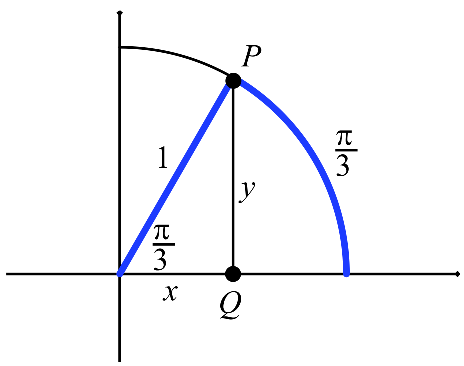


Figure 1.21: The arc $\dfrac{\pi}{3}$ and its associated angle.

Figure 1.18 to conclude that $x = \dfrac{1}{2}$ and $y = \dfrac{\sqrt{3}}{2}$. (Since in this case, $c = 1$.) Therefore, we have just proved that

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Reference Arcs (Reference Angles)

In the beginning activity for this section and in Progress Checks 1.21 and 1.22, we saw that we could relate the coordinates of the terminal point of an arc of length greater than $\dfrac{\pi}{2}$ on the unit circle to the coordinates of the terminal point of an arc of length between 0 and $\dfrac{\pi}{2}$ on the unit circle. This was intended to show that we can do this for any angle of length greater than $\dfrac{\pi}{2}$, and this means that if we know the values of the cosine and sine for any arc (or angle) between 0 and $\dfrac{\pi}{2}$, then we can find the values of the cosine and sine for any arc at all. The arc between 0 and $\dfrac{\pi}{2}$ to which we relate a given arc of length greater than $\dfrac{\pi}{2}$ is called a *reference arc*.

Definition. The **reference arc** $\hat{t}$ (read t-hat) for an arc t is the smallest non-negative arc (always considered non-negative) between the terminal point of the arc t and the closer of the two x-intercepts of the unit circle. Note that the two x-intercepts of the unit circle are $(-1, 0)$ and $(1, 0)$.

The concept of reference arc is illustrated in Figure 1.22. Each of the thicker arcs has length $\hat{t}$ and it can be seen that the coordinates of the points in the second, third, and fourth quadrants are all related to the coordinates of the point in the first quadrant. The signs of the coordinates are all determined by the quadrant in which the point lies.

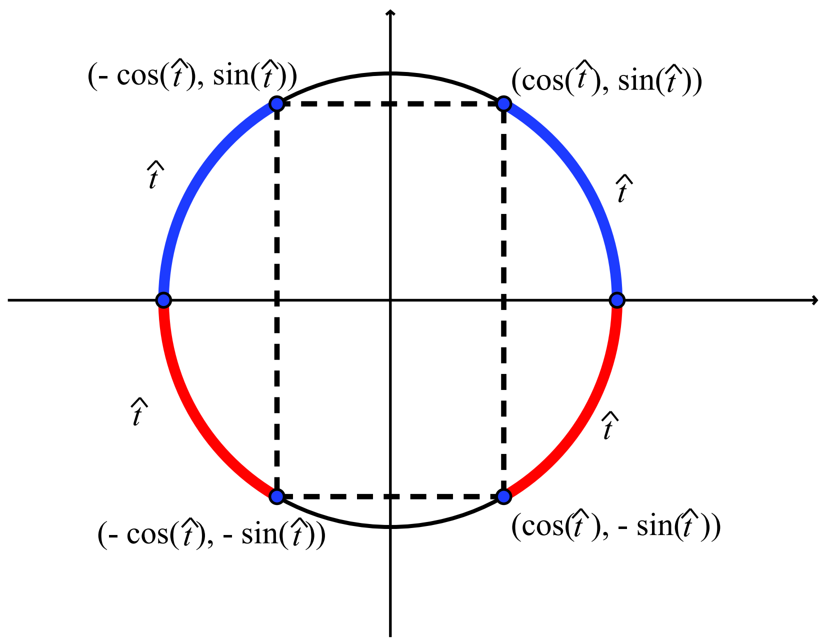


Figure 1.22: Reference arcs.

How we calculate a reference arc for a given arc of length t depends upon the quadrant in which the terminal point of t lies. The diagrams in Figure 1.23 on page 54 illustrate how to calculate the reference arc for an arc of length t with $0 \leq t \leq 2\pi$.

In Figure 1.23, we see that for an arc of length t with $0 \leq t \leq 2\pi$:

- If $\dfrac{\pi}{2} < t < \pi$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(-1, 0)$. So the reference arc is $\pi - t$. In this case, Figure 1.23 shows that

$$\cos(\pi - t) = -\cos(t) \qquad \text{and} \qquad \sin(\pi - t) = \sin(t).$$

- If $\pi < t < \dfrac{3\pi}{2}$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(-1, 0)$. So the reference arc is $t - \pi$. In this case, Figure 1.23 shows that

$$\cos(t - \pi) = -\cos(t) \qquad \text{and} \qquad \sin(t - \pi) = -\sin(t).$$

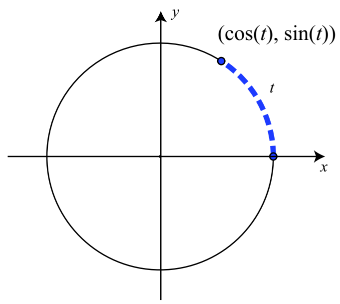
- If $\dfrac{3\pi}{2} < t < 2\pi$, then the point intersecting the unit circle and the x axis that is closest to the terminal point of t is $(1, 0)$. So the reference arc is $2\pi - t$. In this case, Figure 1.23 shows that

$$\cos(2\pi - t) = \cos(t) \qquad \text{and} \qquad \sin(2\pi - t) = -\sin(t).$$

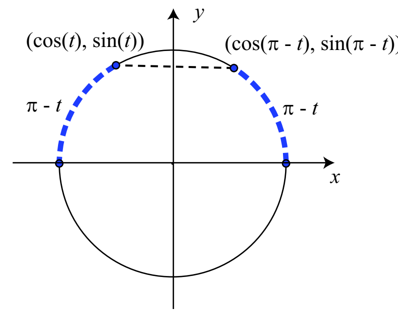
Progress Check 1.23 (Reference Arcs – Part 1)
For each of the following arcs, draw a picture of the arc on the unit circle. Then determine the reference arc for that arc and draw the reference arc in the first quadrant.

1. $t = \dfrac{5\pi}{4}$ **2.** $t = \dfrac{4\pi}{5}$ **3.** $t = \dfrac{5\pi}{3}$

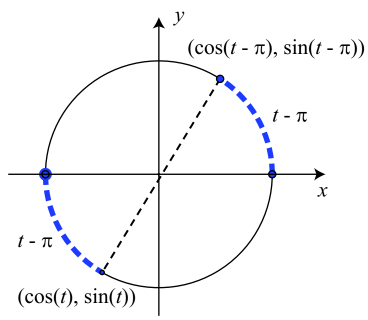
Progress Check 1.24 (Reference Arcs – Part 2)
Although we did not use the term then, in Progress Checks 1.21 and 1.22, we used the facts that $t = \dfrac{\pi}{6}$ and $t = \dfrac{\pi}{4}$ were the reference arcs for other arcs to determine the exact values of the cosine and sine functions for those other arcs. Now use the values of $\cos\left(\dfrac{\pi}{3}\right)$ and $\sin\left(\dfrac{\pi}{3}\right)$ to determine the exact values of the cosine and sine functions for each of the following arcs:



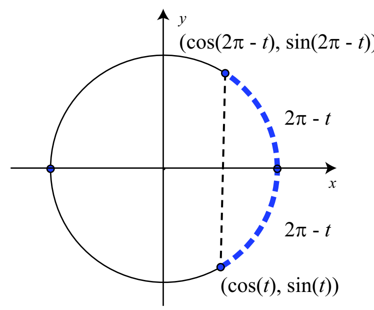
If the arc t is in Quadrant I, then t is its own reference arc.



If the arc t is in Quadrant II, then $\pi - t$ is its reference arc.



If the arc t is in Quadrant III, then $t - \pi$ is its reference arc.



If the arc t is in Quadrant IV, then $2\pi - t$ is its reference arc.

Figure 1.23: Reference arcs

1. $t = \dfrac{2\pi}{3}$ **2.** $t = \dfrac{4\pi}{3}$ **3.** $t = \dfrac{5\pi}{3}$

Reference Arcs for Negative Arcs

Up to now, we have only discussed reference arcs for positive arcs, but the same principles apply when we use negative arcs. Whether the arc t is positive or negative, the reference arc for t is the smallest non-negative arc formed by the terminal point of t and the nearest x-intercept of the unit circle. For example, the arc $t = -\dfrac{\pi}{4}$ is in the fourth quadrant, and the closer of the two x-intercepts of the unit circle is $(1, 0)$. So the reference arc is $\hat{t} = \dfrac{\pi}{4}$ as shown in Figure 1.24.

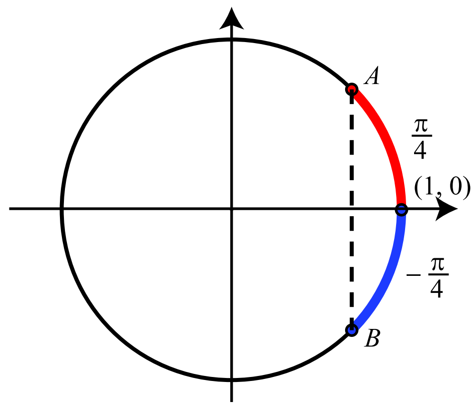


Figure 1.24: Reference Arc for $t = \dfrac{\pi}{4}$.

Since we know that the point A has coordinates $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, we conclude that the point B has coordinates $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$, and so

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Progress Check 1.25 (Reference Arcs for Negative Arcs)
For each of the following arcs, determine the reference arc and the exact values of the cosine and sine functions.

1. $t = -\dfrac{\pi}{6}$

2. $t = -\dfrac{2\pi}{3}$

3. $t = -\dfrac{5\pi}{4}$

Example 1.26 (Using Reference Arcs)
Sometimes we can use the concept of a reference arc even if we do not know the length of the arc but do know the value of the cosine or sine function. For example, suppose we know that

$$0 < t < \frac{\pi}{2} \quad \text{and} \quad \sin(t) = \frac{2}{3}.$$

Are there any conclusions we can make with this information? Following are some possibilities.

1. We can use the Pythagorean identity to determine $\cos(t)$ as follows:

$$\begin{aligned}\cos^2(t) + \sin^2(t) &= 1\\ \cos^2(t) &= 1 - \left(\frac{2}{3}\right)^2\\ \cos^2(t) &= \frac{5}{9}\end{aligned}$$

   Since t is in the first quadrant, we know that $\cos(t)$ is positive, and hence

$$\cos(t) = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

2. Since $0 < t < \dfrac{\pi}{2}$, t is in the first quadrant. Hence, $\pi - t$ is in the second quadrant and the reference arc is t. In the second quadrant we know that the sine is positive, so we can conclude that

$$\sin(\pi - t) = \sin(t) = \frac{2}{3}.$$

Progress Check 1.27 (Working with Reference Arcs)
Following is information from Example 1.26:

$$0 < t < \frac{\pi}{2} \quad \text{and} \quad \sin(t) = \frac{2}{3}.$$

Use this information to determine the exact values of each of the following:

1. $\cos(\pi - t)$
2. $\sin(\pi + t)$
3. $\cos(\pi + t)$
4. $\sin(2\pi - t)$

Summary of Section 1.5

In this section, we studied the following important concepts and ideas:

- The values of $\cos(t)$ and $\sin(t)$ for arcs whose terminal points are on one of the coordinate axes are shown in Table 1.3 below.

- Exact values for the cosine and sine functions at $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, and $\dfrac{\pi}{3}$ are known and are shown in Table 1.3 below.

- A **reference arc** for an arc t is the arc (always considered nonnegative) between the terminal point of the arc t and point intersecting the unit circle and the x-axis closest to it.

- If t is an arc that has an arc $\hat{t}$ as a reference arc, then $|\cos(t)|$ and $|\cos(\hat{t})|$ are the same. Whether $\cos(t) = \cos(\hat{t})$ or $\cos(t) = -\cos(\hat{t})$ is determined by the quadrant in which the terminal side of t lies. The same is true for $\sin(t)$.

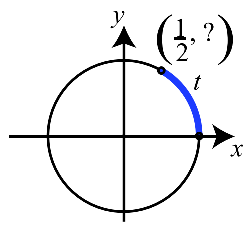
- We can determine the exact values of the cosine and sine functions at any arc with $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$ as reference arc. These arcs between 0 and 2π are shown in Figure 1.17. The results are summarized in Table 1.3 below.

t	$x = \cos(t)$	$y = \sin(t)$	t	$x = \cos(t)$	$y = \sin(t)$
0	1	0	π	-1	0
$\dfrac{\pi}{6}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\dfrac{7\pi}{6}$	$-\dfrac{\sqrt{3}}{2}$	$-\dfrac{1}{2}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{5\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2}$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{4\pi}{3}$	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{3}}{2}$
$\dfrac{\pi}{2}$	0	1	$\dfrac{3\pi}{2}$	0	-1
$\dfrac{2\pi}{3}$	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{5\pi}{3}$	$\dfrac{1}{2}$	$-\dfrac{\sqrt{3}}{2}$
$\dfrac{3\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{7\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2}$
$\dfrac{5\pi}{6}$	$-\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\dfrac{11\pi}{6}$	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{1}{2}$

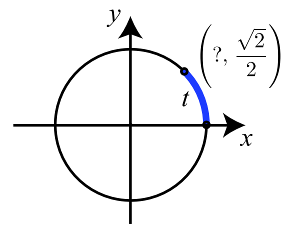
Table 1.3: Exact values of the cosine and sine functions.

Exercises for Section 1.5

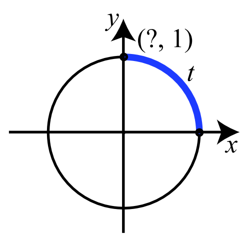
⋆ **1.** A unit circle is shown in each of the following with information about an arc t. In each case, use the information on the unit circle to determine the values of t, $\cos(t)$, and $\sin(t)$.



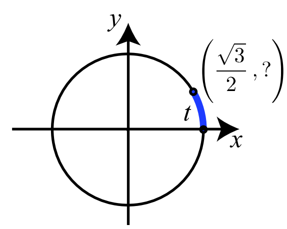
(a)



(c)



(b)



(d)

2. Determine the exact value for each of the following expressions and then use a calculator to check the result. For example,

$$\cos(0) + \sin\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2} \approx 1.8660.$$

⋆ **(a)** $\cos^2\left(\frac{\pi}{6}\right)$

(c) $\dfrac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$

⋆ **(b)** $2\sin^2\left(\frac{\pi}{4}\right) + \cos(\pi)$

(d) $3\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right)$

3. For each of the following, determine the reference arc for the given arc and draw the arc and its reference arc on the unit circle.

⋆ **(a)** $t = \frac{4\pi}{3}$

⋆ **(b)** $t = \frac{13\pi}{8}$

(c) $t = \frac{9\pi}{4}$

⋆ **(d)** $t = -\frac{4\pi}{3}$

(e) $t = -\frac{7\pi}{5}$

(f) $t = 5$

4. For each of the following, draw the given arc t on the unit circle, determine the reference arc for t, and then determine the exact values for $\cos(t)$ and $\sin(t)$.

⋆ **(a)** $t = \dfrac{5\pi}{6}$

(b) $t = \dfrac{5\pi}{4}$

(c) $t = \dfrac{5\pi}{3}$

⋆ **(d)** $t = -\dfrac{2\pi}{3}$

(e) $t = -\dfrac{7\pi}{4}$

(f) $t = \dfrac{19\pi}{6}$

5. (**a**) Use a calculator (in radian mode) to determine five-digit approximations for $\cos(4)$ and $\sin(4)$.

(**b**) Use a calculator (in radian mode) to determine five-digit approximations for $\cos(4 - \pi)$ and $\sin(4 - \pi)$.

(**c**) Use the concept of reference arcs to explain the results in parts (a) and (b).

6. Suppose that we have the following information about the arc t.

$$0 < t < \frac{\pi}{2} \text{ and } \sin(t) = \frac{1}{5}.$$

Use this information to determine the exact values of each of the following:

⋆ **(a)** $\cos(t)$

(b) $\sin(\pi - t)$

(c) $\cos(\pi - t)$

⋆ **(d)** $\sin(\pi + t)$

(e) $\cos(\pi + t)$

(f) $\sin(2\pi - t)$

7. Suppose that we have the following information about the arc t.

$$\frac{\pi}{2} < t < \pi \text{ and } \cos(t) = -\frac{2}{3}.$$

Use this information to determine the exact values of each of the following:

(**a**) $\sin(t)$

(**b**) $\sin(\pi - t)$

(**c**) $\cos(\pi - t)$

(**d**) $\sin(\pi + t)$

(**e**) $\cos(\pi + t)$

(**f**) $\sin(2\pi - t)$

8. Make sure your calculator is in Radian Mode.

(**a**) Use a calculator to find an eight-digit approximation of $\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) = \sin\left(\dfrac{5\pi}{12}\right)$.

(**b**) Determine the **e**xact value of $\sin\left(\dfrac{\pi}{6}\right)+\sin\left(\dfrac{\pi}{4}\right)$.

(**c**) Use a calculator to find an eight-digit approximation of your result in part (b). Compare this to your result in part (a). Does it seem that

$$\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)?$$

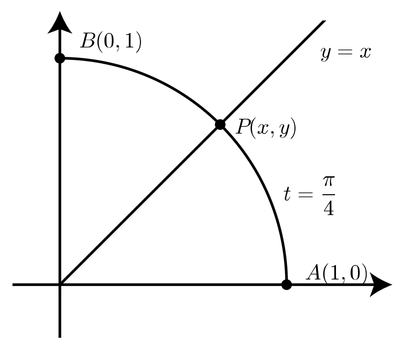
(**d**) Determine the **exact value** of $\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right)+\cos\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right)$.

(**e**) Determine an eight-digit approximation of your result in part (d).

(**f**) Compare the results in parts (a) and (e). Does it seem that

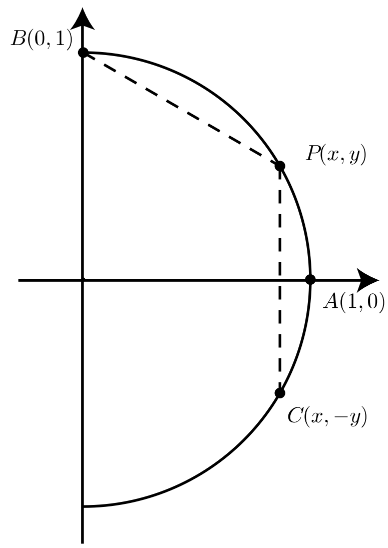
$$\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)?$$

9. This exercise provides an alternate method for determining the exact values of $\cos\left(\dfrac{\pi}{4}\right)$ and $\sin\left(\dfrac{\pi}{4}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \dfrac{\pi}{4}$ on the unit circle. The points $A(1, 0)$ and $B(0, 1)$ are also shown.
Since the point B is the terminal point of the arc of length $\dfrac{\pi}{2}$, we can conclude that the length of the arc from P to B is also $\dfrac{\pi}{4}$. Because of this, we conclude that the point P lies on the line $y = x$ as shown in the diagram. Use this fact to determine the values of x and y. Explain why this proves that



$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

10. This exercise provides an alternate method for determining the exact values of $\cos\left(\frac{\pi}{6}\right)$ and $\sin\left(\frac{\pi}{6}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \frac{\pi}{6}$ on the unit circle. The points $A(1, 0)$, $B(0, 1)$, and $C(x, -y)$ are also shown. Notice that B is the terminal point of the arc $t = \frac{\pi}{2}$, and C is the terminal point of the arc $t = -\frac{\pi}{6}$.



We now notice that the length of the arc from P to B is

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

In addition, the length of the arc from C to P is

$$\frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3}.$$

This means that the distance from P to B is equal to the distance from C to P.

(**a**) Use the distance formula to write a formula (in terms of x and y) for the distance from P to B.

(**b**) Use the distance formula to write a formula (in terms of x and y) for the distance from C to P.

(**c**) Set the distances from (a) and (b) equal to each other and solve the resulting equation for y. To do this, begin by squaring both sides of the equation. In order to solve for y, it may be necessary to use the fact that $x^2 + y^2 = 1$.

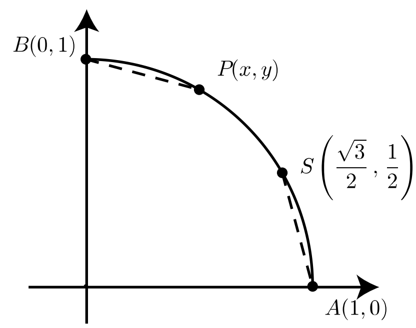
(**d**) Use the value for y in (c) and the fact that $x^2 + y^2 = 1$ to determine the value for x.

Explain why this proves that

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

11. This exercise provides an alternate method for determining the exact values of $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$. The diagram to the right shows the terminal point $P(x, y)$ for an arc of length $t = \frac{\pi}{3}$ on the unit circle. The points $A(1, 0)$, $B(0, 1)$, and $S\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ are also shown. Notice that B is the terminal point of the arc $t = \frac{\pi}{2}$.

From Exercise (10), we know that S is the terminal point of an arc of length $\frac{\pi}{6}$.



We now notice that the length of the arc from A to P is $\frac{\pi}{3}$. In addition, since the length of the arc from A to B is $\frac{\pi}{2}$ and the and the length of the arc from A to P is $\frac{\pi}{3}$, the length of the arc from P to B is equal

$$\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

Since both of the arcs have length $\frac{\pi}{6}$, the distance from A to S is equal to the distance from P to B.

(**a**) Use the distance formula to determine the distance from A to S.

(**b**) Use the distance formula to write a formula (in terms of x and y) for the distance from P to B.

(**c**) Set the distances from (a) and (b) equal to each other and solve the resulting equation for y. To do this, begin by squaring both sides of the equation. In order to solve for y, it may be necessary to use the fact that $x^2 + y^2 = 1$.

(**d**) Use the value for y in (c) and the fact that $x^2 + y^2 = 1$ to determine the value for x.
Explain why this proves that

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

1.6 Other Trigonometric Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How is the tangent function defined? What is the domain of the tangent function?
- What are the reciprocal functions and how are they defined? What are the domains of each of the reciprocal functions?

We defined the cosine and sine functions as the coordinates of the terminal points of arcs on the unit circle. As we will see later, the sine and cosine give relations for certain sides and angles of right triangles. It will be useful to be able to relate different sides and angles in right triangles, and we need other circular functions to do that. We obtain these other circular functions – tangent, cotangent, secant, and cosecant – by combining the cosine and sine together in various ways.

Beginning Activity

Using radian measure:

1. For what values of t is $\cos(t) = 0$?

2. For what values of t is $\sin(t) = 0$?

3. In what quadrants is $\cos(t) > 0$? In what quadrants is $\sin(t) > 0$?

4. In what quadrants is $\cos(t) < 0$? In what quadrants is $\sin(t) < 0$?

The Tangent Function

Next to the cosine and sine, the most useful circular function is the tangent.[3]

[3]The word tangent was introduced by Thomas Fincke (1561-1656) in his *Flenspurgensis Geometriae rotundi libri* XIIII where he used the word tangens in Latin. From *"Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

Definition. The **tangent function** is the quotient of the sine function divided by the cosine function. So the tangent of a real number t is defined to be $\dfrac{\sin(t)}{\cos(t)}$ for those values t for which $\cos(t) \neq 0$. The common abbreviation for the tangent of t is

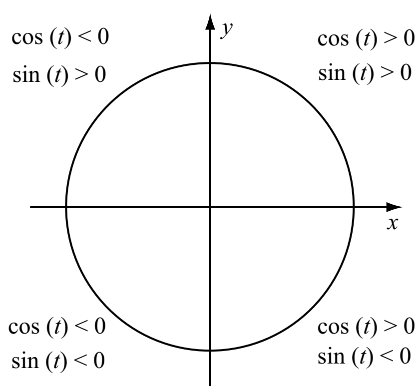
$$\tan(t) = \frac{\sin(t)}{\cos(t)}.$$

In this definition, we need the restriction that $\cos(t) \neq 0$ to make sure the quotient is defined. Since $\cos(t) = 0$ whenever $t = \dfrac{\pi}{2} + k\pi$ for some integer k, we see that $\tan(t)$ is defined when $t \neq \dfrac{\pi}{2} + k\pi$ for all integers k. So

The **domain of the tangent function** is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

Notice that although the domain of the sine and cosine functions is all real numbers, this is not true for the tangent function.

When we worked with the unit circle definitions of cosine and sine, we often used the following diagram to indicate signs of $\cos(t)$ and $\sin(t)$ when the terminal point of the arc t is in a given quadrant.



Progress Check 1.28 (Signs and Values of the Tangent Function)
Considering t to be an arc on the unit circle, for the terminal point of t:

1. In which quadrants is $\tan(t)$ positive?
2. In which quadrants is $\tan(t)$ negative?

3. For what values of t is $\tan(t) = 0$?

4. Complete Table 1.4, which gives the values of cosine, sine, and tangent at the common reference arcs in Quadrant I.

t	$\cos(t)$	$\sin(t)$	$\tan(t)$
0		0	
$\frac{\pi}{6}$		$\frac{1}{2}$	
$\frac{\pi}{4}$		$\frac{\sqrt{2}}{2}$	
$\frac{\pi}{4}$		$\frac{\sqrt{3}}{2}$	
$\frac{\pi}{2}$		1	

Table 1.4: Values of the Tangent Function

Just as with the cosine and sine, if we know the values of the tangent function at the reference arcs, we can find its values at any arc related to a reference arc. For example, the reference arc for the arc $t = \frac{5\pi}{3}$ is $\frac{\pi}{3}$. So

$$\begin{aligned}
\tan\left(\frac{5\pi}{3}\right) &= \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} \\
&= \frac{-\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} \\
&= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= -\sqrt{3}
\end{aligned}$$

We can shorten this process by just using the fact that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and that $\tan\left(\frac{5\pi}{3}\right) < 0$ since the terminal point of the arc $\frac{5\pi}{3}$ is in the fourth quadrant.

$$\tan\left(\frac{5\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}.$$

Progress Check 1.29 (Values of the Tangent Function)

1. Determine the exact values of $\tan\left(\frac{5\pi}{4}\right)$ and $\tan\left(\frac{5\pi}{6}\right)$.

2. Determine the exact values of $\cos(t)$ and $\tan(t)$ if it is known that $\sin(t) = \frac{1}{3}$ and $\tan(t) < 0$.

The Reciprocal Functions

The remaining circular or trigonometric functions are reciprocals of the cosine, sine, and tangent functions. Since these functions are reciprocals, their domains will be all real numbers for which the denominator is not equal to zero. The first we will introduce is the secant function. [4] function.

Definition. The **secant function** is the reciprocal of the cosine function. So the secant of a real number t is defined to be $\frac{1}{\cos(t)}$ for those values t where $\cos(t) \neq 0$. The common abbreviation for the secant of t is

$$\sec(t) = \frac{1}{\cos(t)}.$$

Since the tangent function and the secant function use $\cos(t)$ in a denominator, they have the same domain. So

[4]The term secant was introduced by was by Thomas Fincke (1561-1656) in his Thomae Finkii Flenspurgensis Geometriae rotundi libri XIIII, Basileae: Per Sebastianum Henricpetri, 1583. Vieta (1593) did not approve of the term secant, believing it could be confused with the geometry term. He used Transsinuosa instead. From *"Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

The **domain of the secant function** is the set of all real numbers t for which $t \neq \frac{\pi}{2} + k\pi$ for every integer k.

Next up is the cosecant function. [5]

Definition. The **cosecant function** is the reciprocal of the sine function. So the cosecant of a real number t is defined to be $\frac{1}{\sin(t)}$ for those values t where $\sin(t) \neq 0$. The common abbreviation for the cosecant of t is

$$\csc(t) = \frac{1}{\sin(t)}.$$

Since $\sin(t) = 0$ whenever $t = k\pi$ for some integer k, we see that

The **domain of the cosecant function** is the set of all real numbers t for which $t \neq k\pi$ for every integer k.

Finally, we have the cotangent function. [6]

Definition. The **cotangent function** is the reciprocal of the tangent function. So the cotangent of a real number t is defined to be $\frac{1}{\tan(t)}$ for those values t where $\tan(t) \neq 0$. The common abbreviation for the cotangent of t is

$$\cot(t) = \frac{1}{\tan(t)}.$$

Since $\tan(t) = 0$ whenever $t = k\pi$ for some integer k, we see that

The **domain of the cotangent function** is the set of all real numbers t for which $t \neq k\pi$ for every integer k.

[5] Georg Joachim von Lauchen Rheticus appears to be the first to use the term cosecant (as cosecans in Latin) in his *Opus Palatinum de triangulis*. From *Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

[6] The word cotangent was introduced by Edmund Gunter in *Canon Triangulorum* (Table of Artificial Sines and Tangents) where he used the term cotangens in Latin. From *Earliest Known Uses of Some of the Words of Mathematics* at http://jeff560.tripod.com/mathword.html.

A Note about Calculators

When it is not possible to determine exact values of a trigonometric function, we use a calculator to determine approximate values. However, please keep in mind that most calculators only have keys for the sine, cosine, and tangent functions. With these calculators, we must use the definitions of cosecant, secant, and cotangent to determine approximate values for these functions.

Progress Check 1.30 (Values of Trigonometric Functions
When possible, find the exact value of each of the following functional values. When this is not possible, use a calculator to find a decimal approximation to four decimal places.

1. $\sec\left(\frac{7\pi}{4}\right)$
2. $\csc\left(\frac{-\pi}{4}\right)$
3. $\tan\left(\frac{7\pi}{8}\right)$
4. $\cot\left(\frac{4\pi}{3}\right)$
5. $\csc(5)$

Progress Check 1.31 (Working with Trigonometric Functions

1. If $\cos(x) = \frac{1}{3}$ and $\sin(x) < 0$, determine the exact values of $\sin(x)$, $\tan(x)$, $\csc(x)$, and $\cot(x)$.
2. If $\sin(x) = -\frac{7}{10}$ and $\tan(x) > 0$, determine the exact values of $\cos(x)$ and $\cot(x)$.
3. What is another way to write $(\tan(x))(\cos(x))$?

Summary of Section 1.6

In this section, we studied the following important concepts and ideas:

- The **tangent function** is the quotient of the sine function divided by the cosine function. That is,
$$\tan(t) = \frac{\sin(t)}{\cos(t)},$$
for those values t for which $\cos(t) \neq 0$. The **domain of the tangent function** is the set of all real numbers t for which $t \neq \frac{\pi}{2} + k\pi$ for every integer k.

- The reciprocal functions are the secant, cosecant, and tangent functions.

Reciprocal Function	**Domain**
$\sec(t) = \dfrac{1}{\cos(t)}$	The set of real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.
$\csc(t) = \dfrac{1}{\sin(t)}$	The set of real numbers t for which $t \neq k\pi$ for every integer k.
$\cot(t) = \dfrac{1}{\tan(t)}$	The set of real numbers t for which $t \neq k\pi$ for every integer k.

Exercises for Section 1.6

⋆ **1.** Complete the following table with the exact values of each functional value if it is defined.

t	$\cot(t)$	$\sec(t)$	$\csc(t)$
0			
$\dfrac{\pi}{6}$			
$\dfrac{\pi}{4}$			
$\dfrac{\pi}{3}$			
$\dfrac{\pi}{2}$			

2. Complete the following table with the exact values of each functional value if it is defined.

t	$\cot(t)$	$\sec(t)$	$\csc(t)$
$\dfrac{2\pi}{3}$			
$\dfrac{7\pi}{6}$			
$\dfrac{7\pi}{4}$			
$-\dfrac{\pi}{3}$			
π			

3. Determine the quadrant in which the terminal point of each arc lies based on the given information.

⋆ **(a)** $\cos(x) > 0$ and $\tan(x) < 0$.

⋆ **(b)** $\tan(x) > 0$ and $\csc(x) < 0$.

(c) $\cot(x) > 0$ and $\sec(x) > 0$.

(d) $\sin(x) < 0$ and $\sec(x) > 0$.

(e) $\sec(x) < 0$ and $\csc(x) > 0$.

(f) $\sin(x) < 0$ and $\cot(x) > 0$.

⋆ **4.** If $\sin(t) = \dfrac{1}{3}$ and $\cos(t) < 0$, determine the exact values of $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

5. If $\cos(t) = -\dfrac{3}{5}$ and $\sin(t) < 0$, determine the exact values of $\sin(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

6. If $\sin(t) = -\dfrac{2}{5}$ and $\tan(t) < 0$, determine the exact values of $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

7. If $\sin(t) = 0.273$ and $\cos(t) < 0$, determine the three-digit approximations for $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

8. In each case, determine the arc t that satisfies the given conditions or explain why no such arc exists.

⋆ **(a)** $\tan(t) = 1$, $\cos(t) = -\dfrac{1}{\sqrt{2}}$, and $0 < t < 2\pi$.

⋆ **(b)** $\sin(t) = 1$, $\sec(t)$ is undefined, and $0 < t < \pi$.

(c) $\sin(t) = \dfrac{\sqrt{2}}{2}$, $\sec(t) = -\sqrt{2}$, and $0 < t < \pi$.

(d) $\sec(t) = -\dfrac{2}{\sqrt{3}}$, $\tan(t) = \sqrt{3}$, and $0 < t < 2\pi$.

(e) $\csc(t) = \sqrt{2}$, $\tan(t) = -1$, and $0 < t < 2\pi$.

9. Use a calculator to determine four-digit decimal approximations for each of the following.

(a) $\csc(1)$

(b) $\tan\left(\dfrac{12\pi}{5}\right)$

(c) $\cot(5)$

(d) $\sec\left(\dfrac{13\pi}{8}\right)$

(e) $\sin^2(5.5)$

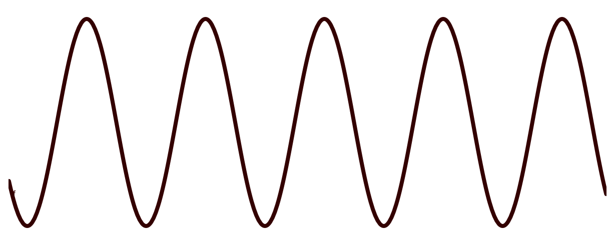
(f) $1 + \tan^2(2)$

(g) $\sec^2(2)$

Chapter 2

Graphs of the Trigonometric Functions

Wherever we live, we have experienced the fact that the amount of daylight where we live varies over the year but that the amount of daylight we be the about the same in a given month, say March, of every year. This is an example of a periodic (or repeating) phenomena. Another example of something that is periodic is a sound wave. In fact, waves are usually represented by a picture such as the following:



As we will see in this Chapter, the sine and cosine functions provide an excellent way to study these waves mathematically.

2.1 Graphs of the Cosine and Sine Functions

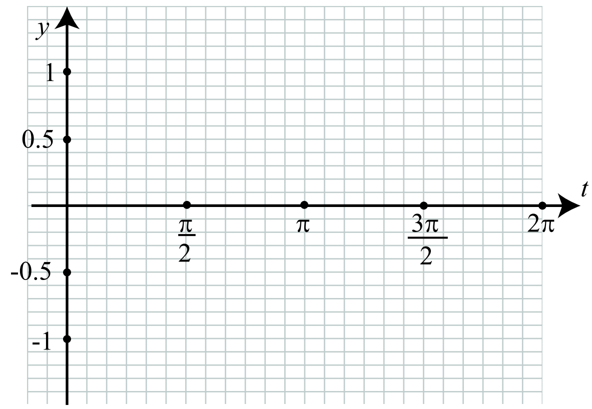
Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the important properties of the graphs of the functions given by $y = \cos(x)$ and $y = \sin(x)$?
- What are the domains of the sine and cosine functions? What are the ranges of the sine and cosine functions?
- What are the periods of the sine and cosine functions? What does period mean?
- What is amplitude? How does the amplitude affect the graph of the sine or cosine?

Beginning Activity

1. The most basic form of drawing the graph of a function is to plot points. Use the values in the given table to plot the points on the graph of $y = \sin(x)$ and then draw the graph of $y = \sin(t)$ for $0 \le t \le 2\pi$. **Note**: On the t-axis, the grid lines are $\dfrac{\pi}{12}$ units apart and on the y-axis, the grid lines are 0.1 of a unit apart.



t	$\sin(t)$	$\sin(t)$ (approx)
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	0.5
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.707
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.866
$\frac{\pi}{2}$	1	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.866
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.714
$\frac{5\pi}{6}$	$\frac{1}{2}$	0.5
π	0	0

t	$\sin(t)$	$\sin(t)$ (approx)
$\frac{7\pi}{6}$	$-\frac{1}{2}$	-0.5
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	-0.707
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	-0.866
$\frac{3\pi}{2}$	-1	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	-0.866
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	-0.707
$\frac{11\pi}{6}$	$-\frac{1}{2}$	-0.5
2π	0	0

2. We can also use a graphing calculator or other graphing device to draw the graph of the sine function. Make sure the device is set to radian mode and use it to draw the graph of $y = \sin(t)$ using $-2\pi \leq t \leq 4\pi$ and $-1.2 \leq y \leq 1.2$. **Note**: Many graphing utilities require the use of x as the independent variable. For such devices, we need to use $y = \sin(x)$. This will make no difference in the graph of the function.

(**a**) Compare this to the graph from part (1). What are the similarities? What are the differences?

(**b**) Find four separate values of t where the graph of the sine function crosses the t-axis. Such values are called t**-intercepts** of the sine function (or **roots** or **zeros**).

(**c**) Based on the graphs, what appears to be the maximum value of $\sin(t)$. Determine two different values of t that give this maximum value of $\sin(t)$.

(**d**) Based on the graphs, what appears to be the minimum value of $\sin(t)$. Determine two different values of t that give this minimum value of $\sin(t)$.

The Periods of the Sine and Cosine Functions

One thing we can observe from the graphs of the sine function in the beginning activity is that the graph seems to have a "wave" form and that this "wave" repeats as we move along the horizontal axis. We see that the portion of the graph between 0 and 2π seems identical to the portion of the graph between 2π and 4π and to the portion of the graph between -2π and 0. The graph of the sine function is exhibiting what is known as a periodic property. Figure 2.1 shows the graph of $y = \sin(t)$ for three cycles.

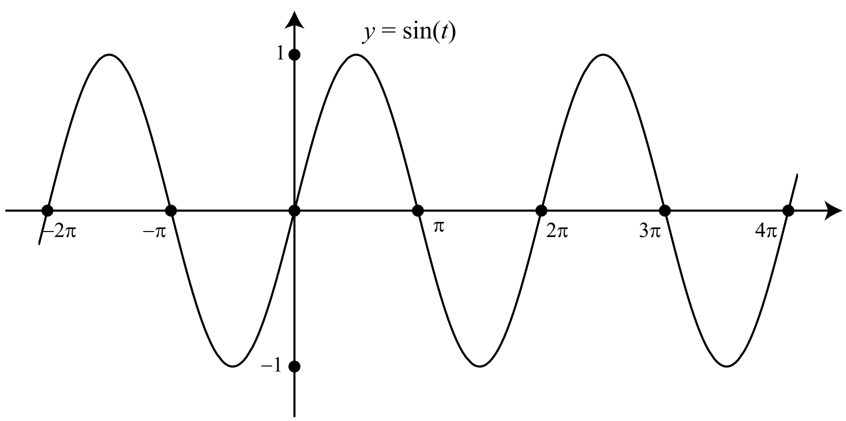


Figure 2.1: Graph of $y = \sin(t)$ with $-2\pi \le t \le 4\pi$

We say that the sine function is a **periodic function**. Such functions are often used to model repetitious phenomena such as a pendulum swinging back and forth, a weight attached to a spring, and a vibrating guitar string.

The reason that the graph of $y = \sin(t)$ repeats is that the value of $\sin(t)$ is the y-coordinate of a point as it moves around the unit circle. Since the circumference of the unit circle is 2π units, an arc of length $(t + 2\pi)$ will have the same terminal point as an arc of length t. Since $\sin(t)$ is the y-coordinate of this point, we see that $\sin(t + 2\pi) = \sin(t)$. This means that the period of the sine function is 2π. Following is a more formal definition of a periodic function.

Definition. A function f is **periodic** with period p if $f(t + p) = f(t)$ for all t in the domain of f and p is the smallest positive number that has this property.

Notice that if f is a periodic function with period p, then if we add $2p$ to t, we get

$$f(t + 2p) = f((t + p) + p) = f(t + p) = f(t).$$

We can continue to repeat this process and see that for any integer k,

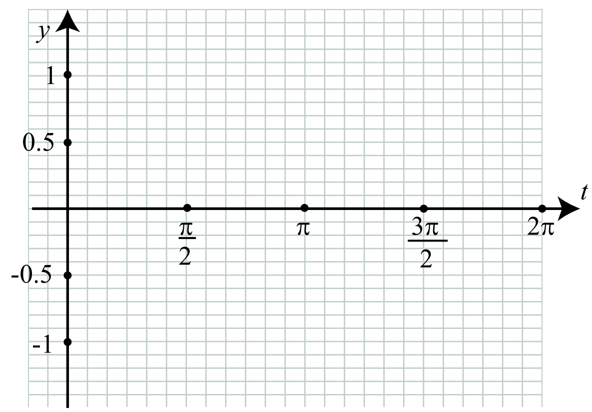
$$f(t + kp) = f(t).$$

So far, we have been discussing only the sine function, but we get similar behavior with the cosine function. Recall that the wrapping function wraps the number line around the unit circle in a way that repeats in segments of length 2π. This is periodic behavior and it leads to periodic behavior of both the sine and cosine functions. Since the value of the sine function is the y-coordinate of a point on the unit circle and the value of the cosine function is the x-coordinate of the same point on the unit circle, the sine and cosine functions repeat every time we make one wrap around the unit circle. That is,

$$\cos(t + 2\pi) = \cos(t) \quad \text{and} \quad \sin(t + 2\pi) = \sin(t).$$

It is important to recognize that 2π is the smallest number that makes this happen. Therefore, the cosine and sine functions are periodic with period 2π.

Progress Check 2.1 (The Graph of the Cosine Function).
We can, of course, use a graphing utility to draw the graph of the cosine function. However, it does help to understand the graph if we actually draw the graph by hand as we did for the sine function in the beginning activity. Use the values in the given table to plot the points on the graph of $y = \cos(t)$ and then draw the graph of $y = \cos(t)$ for $0 \le t \le 2\pi$.



t	$\cos(t)$	$\cos(t)$ (approx)
0	1	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	0.866
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.707
$\frac{\pi}{3}$	$\frac{1}{2}$	0.5
$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$	-0.5
$\frac{3\pi}{4}$	$\frac{-\sqrt{2}}{2}$	-0.714
$\frac{5\pi}{6}$	$\frac{-\sqrt{3}}{2}$	-0.866
π	-1	-1

t	$\cos(t)$	$\cos(t)$ (approx)
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	-0.866
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	-0.714
$\frac{4\pi}{3}$	$-\frac{1}{2}$	-0.5
$\frac{3\pi}{2}$	0	-1
$\frac{2\pi}{3}$	$\frac{1}{2}$	0.5
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.707
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	-0.866
2π	1	1

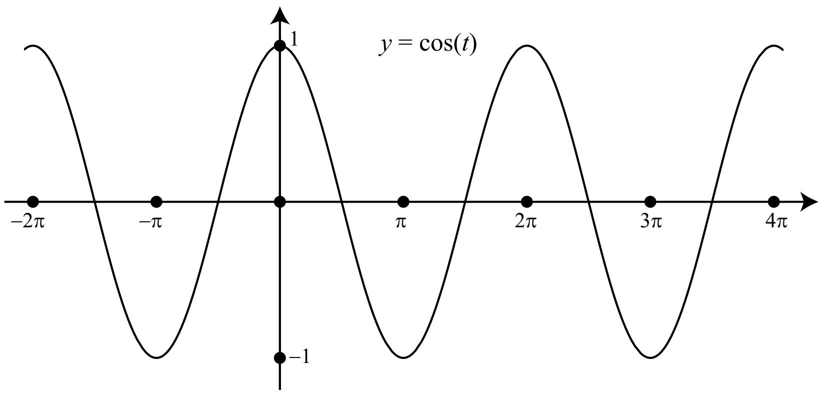


Figure 2.2: Graph of $y = \cos(t)$ with $-2\pi \le t \le 4\pi$

Progress Check 2.2 (The Cosine Function).

1. Compare the graph in Figure 2.2 to the graph from Progress Check 2.1. What are the similarities? What are the differences?
2. Find four separate values of t where the graph of the cosine function crosses the t-axis. Such values are called **t-intercepts** of the cosine function (or **roots** or **zeros**).
3. Based on the graphs, what appears to be the maximum value of $\cos(t)$. Determine two different values of t that give this maximum value of $\cos(t)$.
4. Based on the graphs, what appears to be the minimum value of $\cos(t)$. Determine two different values of t that give this minimum value of $\cos(t)$.

Activity 2.3 (The Graphs of the Sine and Cosine Functions).
We have now constructed the graph of the sine and cosine functions by plotting points and by using a graphing utility. We can have a better understanding of these graphs if we can see how these graphs are related to the unit circle definitions of $\sin(t)$ and $\cos(t)$. We will use two Geogebra applets to help us do this.

The first applet is called *Sine Graph Generator*. The web address is

`http://gvsu.edu/s/Ly`

To begin, just move the slider for t until you get $t = 1$ and observe the resulting image. On the left, there will be a copy of the unit circle with an arc drawn that has length 1. The y-coordinate of the terminal point of this arc (0.84 rounded to the nearest hundredth) will also be displayed. The horizontal line will be connected to the point $(1, 0.84)$ on the graph of $y = \sin(t)$. As the values of t are changed with the slider, more points will be drawn in this manner on the graph of $y = \sin(t)$.

The other applet is called *Cosine Graph Generator* and it works in a manner similar to *Sine Graph Generator*. The web address for this applet is

http://gvsu.edu/s/Lz

Properties of the Graphs of the Sine and Cosine Functions

The graphs of $y = \sin(t)$ and $y = \cos(t)$ are called **sinusoidal waves** and the sine and cosine functions are called **sinusoidal functions**. Both of these particular sinusoidal waves have a period of 2π. The graph over one period is called a **cycle of the graph**. As with other functions in our previous study of algebra, another important property of graphs is their intercepts, in particular, the horizontal intercepts or the points where the graph crosses the horizontal axis. One big difference from algebra is that the sine and cosine functions have infinitely many horizontal intercepts.

In Progress Check 2.2, we used Figure 2.2 and determined that

$$-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

are t-intercepts on the graph of $y = \cos(t)$. In particular,

> In the interval $[0, 2\pi]$, the only t-intercepts of $y = \cos(t)$ are $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

There are, of course, other t-intercepts, and this is where the period of 2π is helpful. We can generate any other t-intercept of $y = \cos(t)$ by adding integer multiples of the period 2π to these two values. For example, if we add 6π to each of them, we see that

$$t = \frac{13\pi}{2} \text{ and } t = \frac{15\pi}{2} \text{ are } t \text{ intercepts of } y = \cos(t).$$

Progress Check 2.4 (The t-intercepts of the Sine Function)
Use a graph to determine the t-intercepts of $y = \sin(t)$ in the interval $[0, 2\pi]$. Then use the period property of the sine function to determine the t-intercepts of $y = \sin(t)$ in the interval $[-2\pi, 4\pi]$. Compare this result to the graph in Figure 2.1. Finally, determine two t-intercepts of $y = \sin(t)$ that are not in the interval $[-2\pi, 4\pi]$.

Activity 2.5 (Exploring Graphs of Sine Functions)
Do one of the following:

1. Draw the graphs of $y = \sin(x)$, $y = \dfrac{1}{2}\sin(x)$ and $y = 2\sin(x)$, $y = -\sin(x)$, and $y = 2\sin(x)$ on the same axes. Make sure your graphing utility is in radian mode and use $-2\pi \le x \le 2\pi$ and $-2.5 \le y \le 2.5$.

2. Use the Geogebra applet *Amplitude of a Sinusoid* at the following web address:

   http://gvsu.edu/s/LM

   The expression for $g(t)$ can be changed but leave it set to $g(t) = \sin(t)$. The slider can be moved to change the value of A and the graph of $y = A\sin(t)$ will be drawn. Explore these graphs by changing the values of A making sure to use negative values of A as well as positive values of A. (It is possible to change this to $g(t) = \cos(t)$ and explore the graphs of $y = A\cos(t)$.)

The Amplitude of Sine and Cosine Functions

The graphs of the functions from Activity 2.5 should have looked like one of the graphs in Figure 2.3. Both graphs are graphs of $y = A\sin(t)$, but the one on the left is for $A > 0$ and the one on the right is for $A < 0$. Note that when $A < 0$, $-A > 0$. Another important characteristic of a sinusoidal wave is the **amplitude**. The amplitude of each of the graphs in Figure 2.3 is represented by the length of the dashed lines, and we see that this length is equal to $|A|$.

Definition. The **amplitude** of a sinusoidal wave is one-half the distance between the maximum and minimum functional values.

$$\text{Amplitude} = \frac{1}{2}\left|(\text{max } y\text{-coordinate}) - (\text{min } y\text{-coordinate})\right|.$$

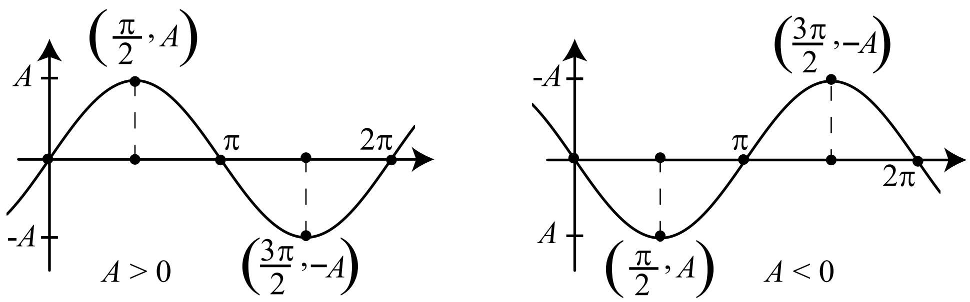


Figure 2.3: Graphs of $y = A\sin(t)$.

Progress Check 2.6 (The Graph of $y = A\cos(t)$)
Draw graphs of $y = A\cos(t)$ for $A > 0$ and for $A < 0$ similar to the graphs for $y = A\sin(t)$ in Figure 2.3.

Using a Graphing Utility

We often will use a graphing utility to draw the graph of a sinusoidal function. When doing so, it is a good idea to use the amplitude to help set an appropriate viewing window. The basic idea is to have the screen on the graphing utility show slightly more than one period of the sinusoid. For example, if we are trying to draw a graph of $y = 3.6\cos(x)$, we could use the following viewing window.

$$-0.5 \le x \le 6.5 \quad \text{and} \quad -4 \le y \le 4.$$

If it is possible, set the x-tickmarks to be every $\dfrac{\pi}{4}$ or $\dfrac{\pi}{2}$ units.

Progress Check 2.7 (Using a Graphing Utility)

1. Use a graphing utility to draw the graph of $y = 3.6\cos(x)$ using the viewing window stated prior to this progress test.
2. Use a graphing utility to draw the graph of $y = -2.75\sin(x)$.

Symmetry and the Negative Identities

Examine the graph of $y = \cos(t)$ shown in Figure 2.2 on page 77. If we focus on that portion of the graph between -2π and 2π, we can notice that the left side of the graph is the "mirror image" of the right side of the graph. To see this better, use the Geogebra applet *Symmetry of the Graph of* $y = \cos(t)$ at the following link:

http://gvsu.edu/s/0t

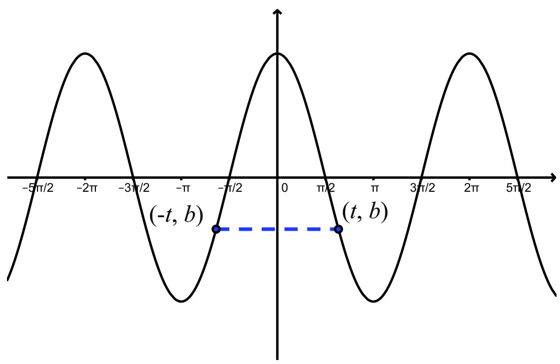


Figure 2.4: Graph Showing Symmetry of $y = \cos(t)$

Figure 2.4 shows a typical image from this applet. Since the second coordinate of a point on the graph is the value of the function at the first coordinate, this figure (and applet) are indicating that $b = \cos(t)$ and $b = \cos(-t)$. That is, this is illustrating the fact that $\cos(-t) = \cos(t)$. The next activity provides an explanation as to why this is true.

Activity 2.8 (Positive and Negative Arcs)
For this activity, we will use the Geogebra applet called *Drawing a Positive Arc and a Negative Arc on the Unit Circle*. A link to this applet is

http://gvsu.edu/s/0l

As the slider for t in the applet is used, an arc of length t will be drawn in blue and an arc of length $-t$ will be drawn in red. In addition, the coordinates of the terminal points of both the arcs t and $-t$ will be displayed. Study the coordinates of these two points for various values of t. What do you observe? Keeping in mind that the coordinates of these points can also be represented as

$$(\cos(t), \sin(t)) \text{ and } (\cos(-t), \sin(-t)),$$

what does these seem to indicate about the relationship between $\cos(-t)$ and $\cos(t)$? What about the relationship between $\sin(-t)$ and $\sin(t)$?

Figure 2.5 shows a typical situation illustrated in Activity 2.8. A positive arc and it corresponding negative arc have been drawn on the unit circle. What we

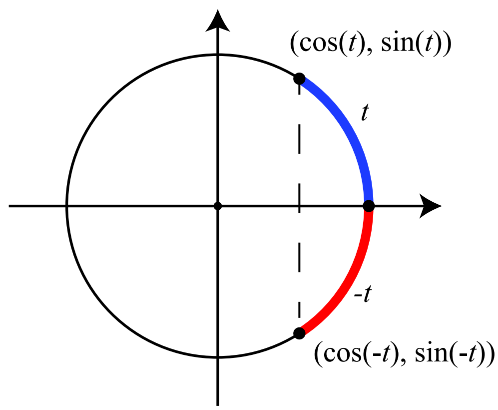


Figure 2.5: An Arc and a Negative Arc on the Unit Circle

have seen is that if the terminal point of the arc t is (a, b), then by the symmetry of the circle, we see that the terminal point of the arc $-t$ is $(a, -b)$. So the diagram illustrates the following results, which are sometimes called **negative arc identities**.

Negative Arc Identities
For every real number t,

$$\sin(-t) = -\sin(t) \qquad\qquad \cos(-t) = \cos(t).$$

To further verify the negative arc identities for sine and cosine, use a graphing utility to:

- Draw the graph of $y = \cos(-x)$ using $0 \leq x \leq 2\pi$. The graph should be identical to the graph of $y = \cos(x)$.

- Draw the graph of $y = -\sin(-x)$ using $0 \leq x \leq 2\pi$. The graph should be identical to the graph of $y = \sin(x)$.

These so-called negative arc identities give us a way to look at the symmetry of the graphs of the cosine and sine functions. We have already illustrated the symmetry of the cosine function in Figure 2.4. Because of this, we say the graph of $y = \cos(t)$ is **symmetric about the** y**-axis**.

What about symmetry in the graph of the sine function? Figure 2.6 illustrates what the negative identity $\sin(-t) = -\sin(t)$ implies about the symmetry of $y = \sin(t)$. In this case, we say the graph of $y = \sin(t)$ is **symmetric about the origin**.

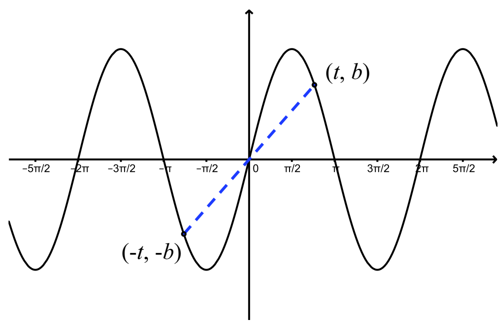


Figure 2.6: Graph Showing Symmetry of $y = \sin(t)$.

To see the symmetry of the graph of the sine function better, use the Geogebra applet *Symmetry of the Graph of* $y = \sin(t)$ at the following link:

http://gvsu.edu/s/Ou

Summary of Section 2.1

In this section, we studied the following important concepts and ideas:

- The important characteristics of sinusoidal functions of the form $y = A\sin(t)$ or $y = A\cos(t)$ shown in Table 2.1.
- The information in Table 2.1 can seem like a lot to remember, and in fact, in the next sections, we will get a lot more information about sinusoidal waves. So instead of trying to remember everything in Table 2.1, it is better to remember the basic shapes of the graphs as shown in Figure 2.7 and Figure 2.8.

$y = A\sin(t)$		$y = A\cos(t)$
All real numbers	domain	All real numbers
2π	period	2π
$\lvert A\rvert$	amplitude	$\lvert A\rvert$
$(0, 0)$	y-intercept	$(0, A)$
$t = 0$ and $t = \pi$	t-intercepts in $[0, 2\pi)$	$t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$
$\lvert A\rvert$	maximum value	$\lvert A\rvert$
$-\lvert A\rvert$	minimum value	$-\lvert A\rvert$
The interval $[-\lvert A\rvert, \lvert A\rvert]$	range	The interval $[-\lvert A\rvert, \lvert A\rvert]$
$t = \dfrac{\pi}{2}$	when $A > 0$, maximum occurs at	$t = 0$
$t = \dfrac{3\pi}{2}$	when $A > 0$, minimum occurs at	$t = \pi$
$t = \dfrac{3\pi}{2}$	when $A < 0$, maximum occurs at	$t = \pi$
$t = \dfrac{\pi}{2}$	when $A < 0$, minimum occurs at	$t = 0$
the origin	symmetry with respect to	the y-axis

Table 2.1: Characteristics of Sinusoidal Functions.

- One way to remember the location of the tick-marks on the t-axis is to remember the spacing for these tick-marks is one-quarter of a period and the period is 2π. So the spacing is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

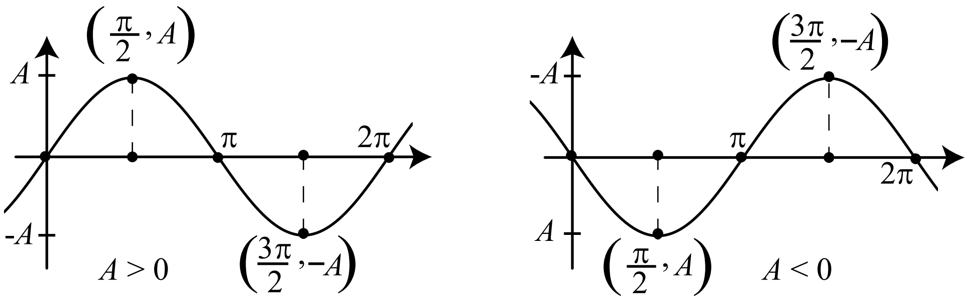


Figure 2.7: Graphs of $y = A\sin(t)$.

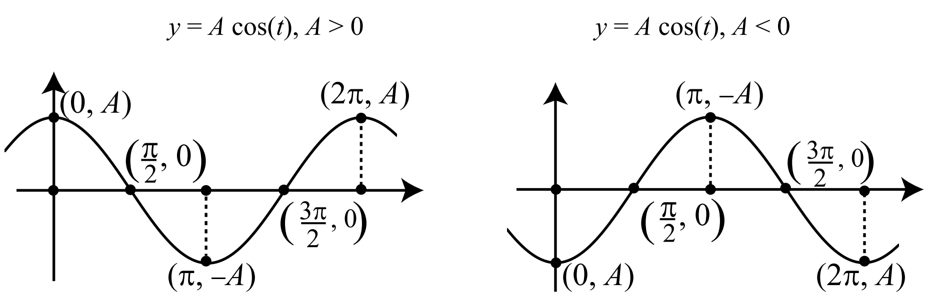


Figure 2.8: Graphs of $y = A\cos(t)$.

Supplemental Material – Even and Odd Functions

There is a more general mathematical context for these types of symmetry, and that has to do with what are called even functions and odd functions.

Definition.

- A function f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f.

So with these definitions, we can say that the cosine function is an even function and the sine function is an odd function. Why do we use these terms? One explanation is that the concepts of even functions and odd functions are used to describe functions f of the form $f(x) = x^n$ for some positive integer n, and the graphs of these functions exhibit different types of symmetry when n is even versus when n is odd.

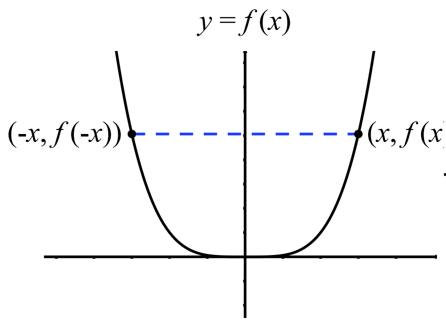


Figure 2.9: $f(x) = x^n$, n even and $f(-x) = f(x)$.

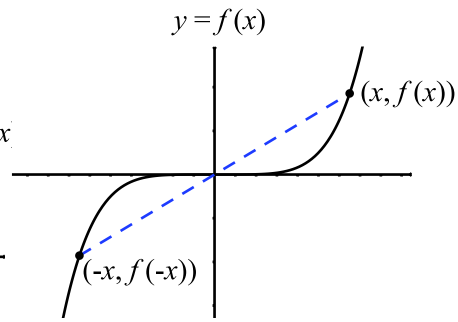


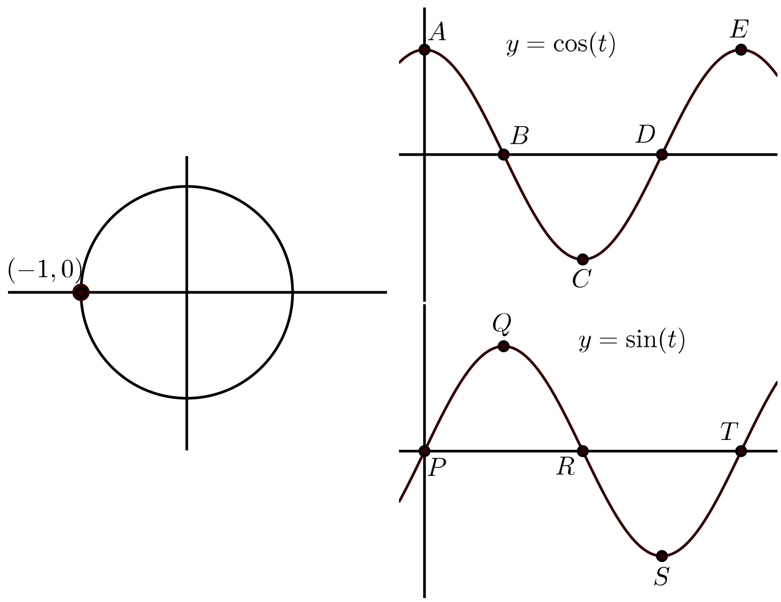
Figure 2.10: $f(x) = x^n$, n odd and $f(-x) = -f(x)$.

In Figure 2.9, we see that when n is even, $f(-x) = f(x)$ since $(-x)^n = x^n$. So the graph is symmetric about the y-axis. When n is odd as in Figure 2.10, $f(-x) = -f(x)$ since $(-x)^n = -x^n$. So the graph is symmetric about the origin. This is why we use the term even functions for those functions f for which $f(-x) = f(x)$, and we use the term odd functions for those functions for which $f(-x) = -f(x)$.

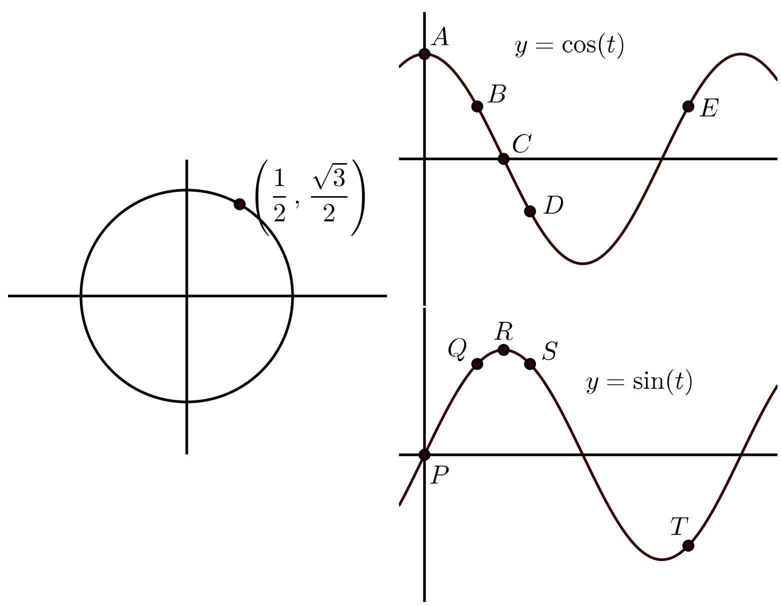
Exercises for Section 2.1

1. In each of the following, the graph on the left shows the terminal point of an arc t (with $0 \leq t < 2\pi$) on the unit circle. The graphs on the right show the graphs of $y = \cos(t)$ and $y = \sin(t)$ with some points on the graph labeled. Match the point on the graphs of $y = \cos(t)$ and $y = \sin(t)$ that correspond to the point on the unit circle. In addition, state the coordinates of the points on $y = \cos(t)$ and $y = \sin(t)$.

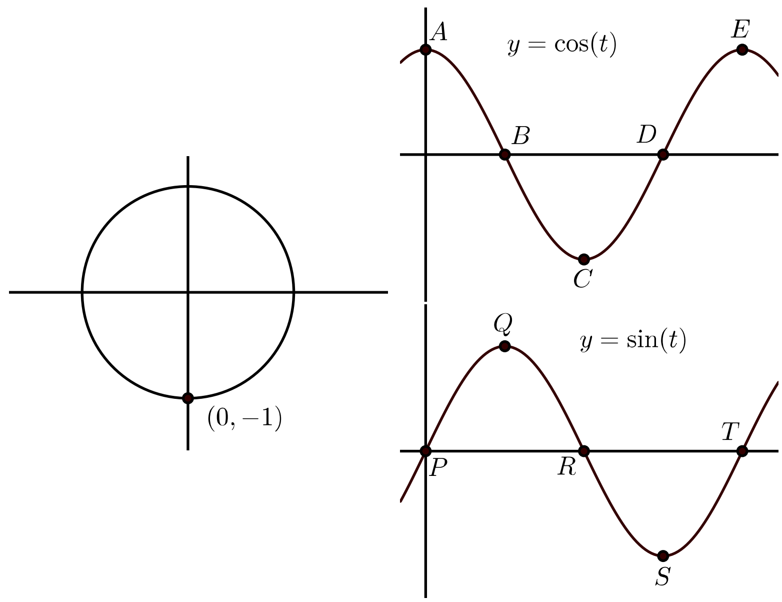
⋆ **(a)**



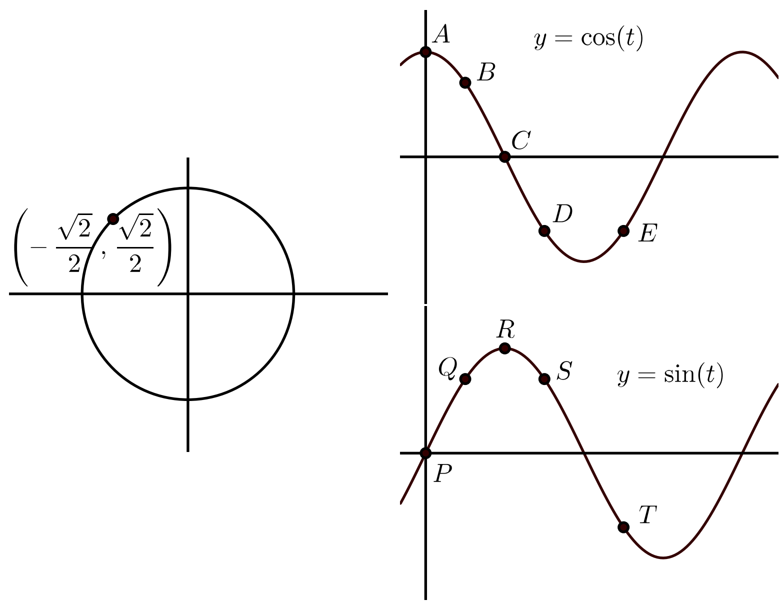
⋆ **(b)**



(c)

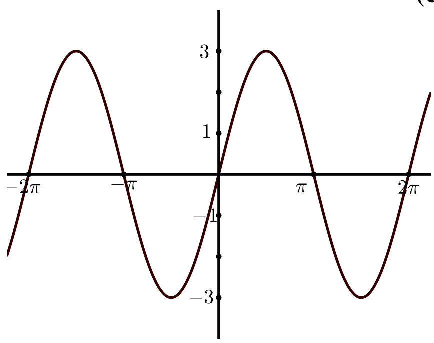


(d)

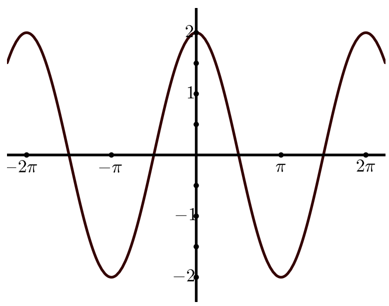


2. For each of the following, determine an equation of the form $y = A\cos(x)$ or $y = A\sin(x)$ for the given graph.

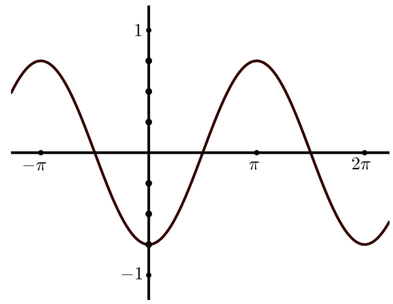
⋆ **(a)**



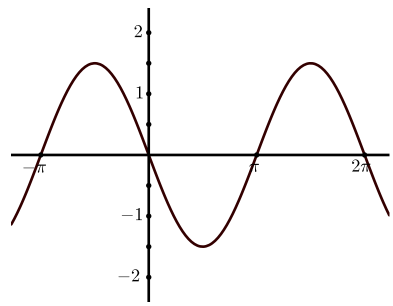
⋆ **(b)**



(c)



(d)



3. Draw the graph of each of the following sinusoidal functions over the indicated interval. For each graph,

- State the t-intercepts on the given interval.
- State the y-intercept.
- State the maximum value of the function and the coordinates of all the points where the maximum value occurs.
- State the minimum value of the function and the coordinates of all the points where the minimum value occurs.

⋆ **(a)** $y = \sin(t)$ with $-2\pi \leq t \leq 2\pi$.

⋆ **(b)** $y = 3\cos(t)$ with $-\pi \leq t \leq 3\pi$.

(c) $y = 5\sin(t)$ with $0 \leq t \leq 4\pi$.

(d) $y = \dfrac{3}{7}\cos(t)$ with $-\pi \leq t \leq 3\pi$.

(e) $y = -2.35\sin(t)$ with $-\pi \leq t \leq \pi$.

(f) $y = -4\cos(t)$ with $0 \leq t \leq 6\pi$.

2.2 Graphs of Sinusoidal Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

Let A, B, C, and D be constants with $B > 0$ and consider the graph of $f(t) = A\sin(B(t - C)) + D$ or $f(t) = A\cos(B(t - C)) + D$.

- How does the value of A affect the graph of f? How is A related to the amplitude of f?
- How does the value of B affect the graph of f? How is B related to the period of f?
- How does the value of C affect the graph of f?
- How does the value of D affect the graph of f?
- What does a phase shift do to a sine or cosine graph? How do we recognize a phase shift from the equation of the sinusoid?
- How do we accurately draw the graph of $y = A\sin(B(t - C)) + D$ or $y = A\cos(B(t-C)) + D$ without a calculator and how do we correctly describe the effects of the constants A, B, C, and D on the graph?

Beginning Activity

In this section, we will study the graphs of functions whose equations are $y = A\sin(B(t - C)) + D$ and $y = A\cos(B(t - C)) + D$ where A, B, C, and D are real number constants. These functions are called **sinusoidal functions** and their graphs are called **sinusoidal waves**. We will first focus on functions whose equations are $y = \sin(Bt)$ and $y = \cos(Bt)$. Now complete Part 1 or Part 2 of this beginning activity.

Part 1 – Using a Geogebra Applet

To begin our exploration, we will use a Geogebra applet called *Period of a Sinusoid*. The web address for this applet is http://gvsu.edu/s/LY

After you open the applet, notice that there is an input box at the top of the screen where you can input a function. For now, leave this set at $g(t) = \sin(t)$. The graph of the sine function should be displayed. The slider at the top can be used to change the value of B. When this is done, the graph of $y = A\sin(Bt)$ will be displayed for the current value of B along with the graph of $y = \sin(t)$.

1. Use the slider to change the value of B. Explain in detail the difference between the graph of $y = g(t) = \sin(t)$ and $y = f(t) = \sin(Bt)$ for a constant $B > 0$. Pay close attention to the graphs and determine the period when

   (a) $B = 2$. **(b)** $B = 3$. **(c)** $B = 4$. **(d)** $B = 0.5$.

   In particular, how does the period of $y = \sin(Bt)$ appear to depend on B? **Note**: Consider doing two separate cases: one when $B > 1$ and the other when $0 < B < 1$.

2. Now click on the reset button in the upper right corner of the screen. This will reset the value of the B to its initial setting of $B = 1$.

3. Change the function to $g(t) = \cos(t)$ and repeat part (1) for the cosine function. Does changing the value of B affect the graph of $y = \cos(Bt)$ in the same way that changing the value for B affects the graph of $y = \sin(Bt)$?

Part 2 – Using a Graphing Utility

Make sure your graphing utility is set to radian mode. **Note**: Most graphing utilities require the use of x (or X) as the independent variable (input) for a function. We will use x for the independent variable when we discuss the use of a graphing utility.

1. We will first examine the graph of $y = \sin(Bx)$ for three different values of B. Graph the three functions:

   $y = \sin(x)$ $\quad$ $y = \sin(2x)$ $\quad$ $y = \sin(4x)$

   using the following settings for the viewing window: $0 \leq x \leq 4\pi$ and $-1.5 \leq y \leq 1.5$. If possible on your graphing utility, set it so that the tickmarks on the x-axis are space at $\dfrac{\pi}{2}$ units. Examine these graphs closely and determine period for each sinusoidal wave. In particular, how does the period of $y = \sin(Bx)$ appear to depend on B?

2. Clear the graphics screen. We will now examine the graph of $y = \sin(Bx)$ for three different values of B. Graph the following three functions:

$$y = \sin(x) \qquad y = \sin\left(\frac{1}{2}x\right) \qquad y = \sin\left(\frac{1}{4}x\right)$$

using the following settings for the viewing window: $0 \leq x \leq 4\pi$ and $-1.5 \leq y \leq 1.5$. If possible on your graphing utility, set it so that the tickmarks on the x-axis are spaced at $\dfrac{\pi}{2}$ units. Examine these graphs closely and determine period for each sinusoidal wave. In particular, how does the period of $y = \sin(Bx)$ appear to depend on B?

3. How does the graph of $y = \sin(Bx)$ appear to be related to the graph of $y = \sin(x)$. **Note**: Consider doing two separate cases: one when $B > 1$ and the other when $0 < B < 1$.

The Period of a Sinusoid

When we discuss an expression such as $\sin(t)$ or $\cos(t)$, we often refer to the expression inside the parentheses as the *argument* of the function. In the beginning activity, we examined situations in which the argument was Bt for some number B. We also saw that this number affects the period of the sinusoid. If we examined graphs close enough, we saw that the period of $y = \sin(Bt)$ and $y = \cos(Bt)$ is equal to $\dfrac{2\pi}{B}$. The graphs in Figure 2.11 illustrate this.

Notice that the graph of $y = \sin(2t)$ has one complete cycle over the interval $[0, \pi]$ and so its period is $\pi = \dfrac{2\pi}{2}$. The graph of $y = \sin(4t)$ has one complete cycle over the interval $\left[0, \dfrac{\pi}{2}\right]$ and so its period is $\dfrac{\pi}{2} = \dfrac{2\pi}{4}$. In these two cases, we had $B > 1$ in $y = \sin(Bt)$. Do we get the same result when $0 < B < 1$? Figure 2.12 shows graphs for $y = \sin\left(\dfrac{1}{2}t\right)$ and $y = \sin\left(\dfrac{1}{4}t\right)$.

Notice that the graph of $y = \sin\left(\dfrac{1}{2}t\right)$ has one complete cycle over the interval $[0, 4\pi]$ and so

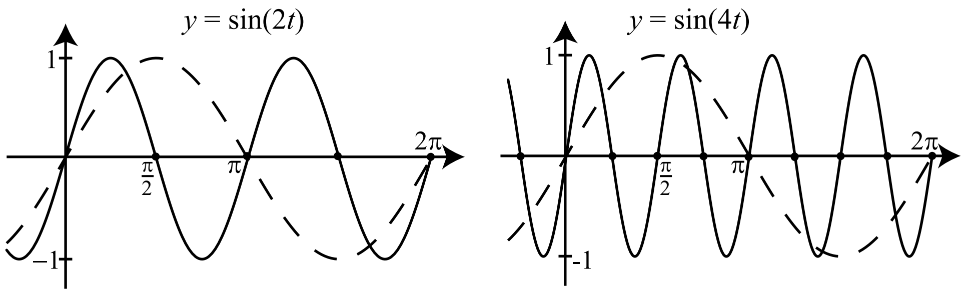


Figure 2.11: Graphs of $y = \sin(2t)$ and $y = \sin(4t)$. The graph of $y = \sin(t)$ is also shown (dashes).

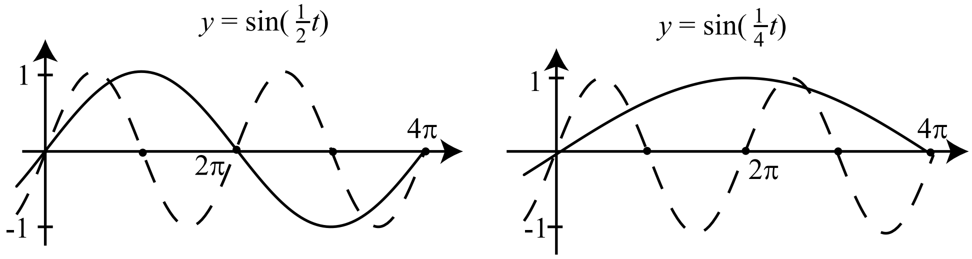


Figure 2.12: Graphs of $y = \sin\left(\frac{1}{2}t\right)$ and $y = \sin\left(\frac{1}{4}t\right)$. The graph of $y = \sin(t)$ is also shown (dashes).

$$\text{the period of } y = \sin\left(\frac{1}{2}t\right) \text{ is } 4\pi = \frac{2\pi}{\frac{1}{2}}.$$

The graph of $y = \sin\left(\frac{1}{4}t\right)$ has one-half of a complete cycle over the interval $[0, 4\pi]$ and so

$$\text{the period of } y = \sin\left(\frac{1}{4}t\right) \text{ is } 8\pi = \frac{2\pi}{\frac{1}{4}}.$$

A good question now is, "Why are the periods of $y = \sin(Bt)$ and $y = \cos(Bt)$ equal to $\frac{2\pi}{B}$?" The idea is that when we multiply the independent variable t by a constant B, it can change the input we need to get a specific output. For example, the input of $t = 0$ in $y = \sin(t)$ and $y = \sin(Bt)$ yield the same output. To complete one period in $y = \sin(t)$ we need to go through interval of length 2π so that our input is 2π. However, in order for the argument (Bt) in $y = \sin(Bt)$ to be 2π, we need $Bt = 2\pi$ and if we solve this for t, we get $t = \dfrac{2\pi}{B}$. So the function given by $y = \sin(Bt)$ (or $y = \cos(Bt)$) will complete one complete cycle when t varies from $t = 0$ to $t = \dfrac{2\pi}{B}$, and hence, the period is $\dfrac{2\pi}{B}$. Notice that if we use $y = A\sin(Bt)$ or $y = A\cos(Bt)$, the value of A only affects the amplitude of the sinusoid and does not affect the period.

> If A is a real number and B is a positive real number, then the period of the functions given by $y = A\sin(Bt)$ and $y = A\cos(Bt)$ is $\dfrac{2\pi}{B}$.

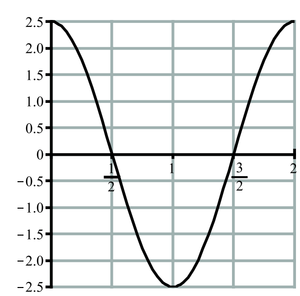
Progress Check 2.9 (The Amplitude and Period of a Sinusoid)

1. Determine the amplitude and period of the following sinusoidal functions.:

(**a**) $y = 3\cos\left(\dfrac{1}{3}t\right)$.

(**b**) $y = -2\sin\left(\dfrac{\pi}{2}t\right)$.

2. The graph to the right is a graph of a sinusoidal function. Determine an equation for this function.



Phase Shift

We will now investigate the effect of subtracting a constant from the argument (*independent variable*) of a circular function. That is, we will investigate what effect the value of a real number C has on the graph of $y = \sin(t - C)$ and $y = \cos(t - C)$.

Activity 2.10 (The Graph of $y = \sin(t - C)$)
Complete Part 1 or Part 2 of this activity.

Part 1 – Using a Geogebra Applet

We will use a Geogebra applet called *Sinusoid – Phase Shift*. The web address for this applet is

http://gvsu.edu/s/Mu

After you open the applet, notice that there is an input box at the top of the screen where you can input a function. For now, leave this set at $g(t) = \sin(t)$. The graph of the sine function should be displayed. The slider at the top can be used to change the value of C. When this is done, the graph of $y = A\sin(t - C)$ will be displayed for the current value of C along with the graph of $y = \sin(t)$.

1. Use the slider to change the value of C. Explain in detail the difference between the graph of $y = g(t) = \sin(t)$ and $y = f(t) = \sin(t - C)$ for a constant C. Pay close attention to the graphs and determine the horizontal shift when

   (a) $C = 1$. **(c)** $C = 3$. **(e)** $C = -2$.
   (b) $C = 2$. **(d)** $C = -1$. **(f)** $C = -3$.

   In particular, describe the difference between the graph of $y = \sin(t - C)$ and the graph of $y = \sin(t)$? **Note**: Consider doing two separate cases: one when $C > 0$ and the other when $C < 0$.

2. Now click on the reset button in the upper right corner of the screen. This will reset the value of the C to its initial setting of $C = 0$.

3. Change the function to $g(t) = \cos(t)$ and repeat part (1) for the cosine function. Does changing the value of C affect the graph of $y = \cos(t - C)$ in the same way that changing the value for C affects the graph of $y = \sin(t - C)$?

Part 2 – Using a Graphing Utility

Make sure your graphing utility is set to radian mode.

1. We will first examine the graph of $y = \sin(x - C)$ for two different values of C. Graph the three functions:

$$y = \sin(x) \qquad y = \sin(x - 1) \qquad y = \sin(x - 2)$$

using the following settings for the viewing window: $0 \le x \le 4\pi$ and $-1.5 \le y \le 1.5$. Examine these graphs closely and describe the difference between the graph of $y = \sin(x - C)$ and the graph of $y = \sin(x)$ for these values of C.

2. Clear the graphics screen. We will now examine the graph of $y = \sin(x - C)$ for two different values of C. Graph the following three functions:

$$y = \sin(x) \qquad y = \sin(x+1) = \sin(x - (-1))$$
$$y = \sin(x+2) = \sin(x - (-2))$$

using the following settings for the viewing window: $-2\pi \le x \le 2\pi$ and $-1.5 \le y \le 1.5$. Examine these graphs closely and describe the difference between the graph of $y = \sin(t - C)$ and the graph of $y = \sin(t)$ for these values of C.

3. Describe the difference between the graph of $y = \sin(x - C)$ and the graph of $y = \sin(x)$? **Note**: Consider doing two separate cases: one when $C > 0$ and the other when $C < 0$.

By exploring the graphs in Activity 2.10, we should notice that when $C > 0$, the graph of $y = \sin(t - C)$ is the graph of $y = \sin(t)$ horizontally translated to the right by C units. In a similar manner, the graph of $y = \cos(t - C)$ is the graph of $y = \cos(t)$ horizontally translated to the right by C units. When working with a sinusoidal graph, such a horizontal translation is called a **phase shift**. This is illustrated in Figure 2.13 , which shows the graphs of $y = \sin(t - 1)$ and $y = \sin\left(t - \frac{\pi}{2}\right)$. For reference, the graph of $y = \sin(t)$ is also shown.

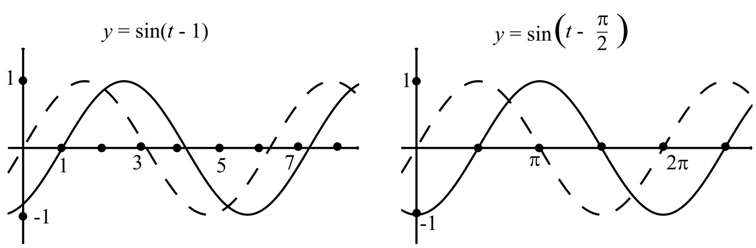


Figure 2.13: Graphs of $y = \sin(t - 1)$ and $y = \sin\left(t - \frac{\pi}{2}\right)$. The graph of $y = \sin(t)$ is also shown (dashes).

So, why are we seeing this phase shift? The reason is that the graph of $y = \sin(t)$ will go through one complete cycle over the interval defined by $0 \leq t \leq 2\pi$. Similarly, the graph of $y = \sin(t - C)$ will go through one complete cycle over the interval defined by $0 \leq t - C \leq 2\pi$. Solving for t, we see that $C \leq t \leq 2\pi + C$. So we see that this cycle for $y = \sin(t)$ has been shifted by C units.

This argument also works when $C < 0$ and when we use the cosine function instead of the sine function. Figure 2.14 illustrates this with $y = \cos(t - (-1))$ and $y = \cos\left(t - \left(-\frac{\pi}{2}\right)\right)$. Notice that we can rewrite these two equations as follows:

$$y = \cos(t - (-1)) \qquad\qquad y = \cos\left(t - \left(-\frac{\pi}{2}\right)\right)$$
$$y = \cos(t + 1) \qquad\qquad y = \cos\left(t + \frac{\pi}{2}\right)$$

We summarize the results for phase shift as follows:

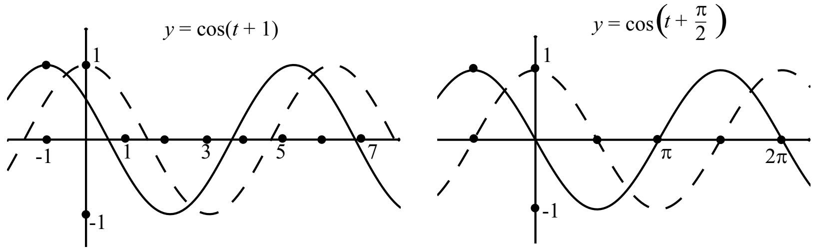


Figure 2.14: Graphs of $y = \cos(t + 1)$ and $y = \cos\left(t + \frac{\pi}{2}\right)$. The graph of $y = \cos(t)$ is also shown (dashes).

For $y = \sin(t - C)$ and $y = \cos(t - C)$, where C is any nonzero real number:

- The graph of $y = \sin(t)$ (or $y = \cos(t)$) is shifted horizontally $|C|$ units. This is called the **phase shift** of the sinusoid.
- If $C > 0$, the graph of $y = \sin(t)$ (or $y = \cos(t)$) is shifted horizontally C units to the right. That is, there is a phase shift of C units to the right.
- If $C < 0$, the graph of $y = \sin(t)$ (or $y = \cos(t)$) is shifted horizontally C units to the left. That is, there is a phase shift of C units to the left.

Progress Check 2.11 (Phase Shift of a Sinusoid)

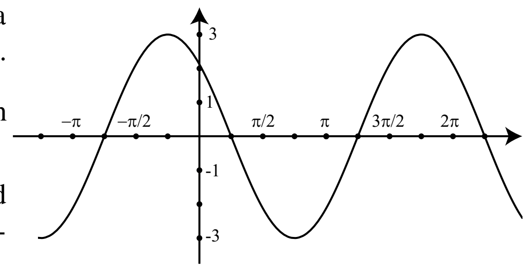
1. Determine the amplitude and phase shift of the following sinusoidal functions.

   (**a**) $y = 3.2\sin\left(t - \dfrac{\pi}{3}\right)$ (**b**) $y = 4\cos\left(t + \dfrac{\pi}{6}\right)$

2. The graph to the right is a graph of a sinusoidal function.

   (**a**) Determine an equation for this function.

   (**b**) Determine a second equation for this function.



Vertical Shift

We have one more transformation of a sinusoid to explore, the so-called vertical shift. This is one by adding a constant to the equation for a sinusoid and is explored in the following activity.

Activity 2.12 (The Vertical Shift of a Sinusoid)
Complete Part 1 or Part 2 of this activity.

Part 1 – Using a Geogebra Applet

We will now investigate the effect of adding a constant to a sinusoidal function. That is, we will investigate what effect the value of a real number D has the graph of $y = A\sin(B(t - C)) + D$ and $y = A\cos(B(t - C)) + D$. We will use a Geogebra applet called *Exploring a Sinusoid*. The web address for this applet is

http://gvsu.edu/s/LX

After you open the applet, notice that there is an input box at the top of the screen where you can input a function. For now, leave this set at $g(t) = \sin(t)$. The graph of the sine function should be displayed. There are four sliders at the top that can be used to change the values of A, B, C, and D.

1. Leave the values $A = 1$, $B = 1$, and $C = 0$ set. Use the slider for D to change the value of D. When this is done, the graph of $y = \sin(t) + D$ will be displayed for the current value of D along with the graph of $y = \sin(t)$.

(**a**) Use the slider to change the value of D. Explain in detail the difference between the graph of $y = g(t) = \sin(t)$ and $y = f(t) = \sin(t) + D$ for a constant D. Pay close attention to the graphs and determine the vertical shift when

i. $D = 1$.	iii. $D = 3$.	v. $D = -2$.
ii. $D = 2$.	iv. $D = -1$.	vi. $D = -3$.

In particular, describe the difference between the graph of $y = \sin(t) + D$ and the graph of $y = \sin(t)$? **Note**: Consider doing two separate cases: one when $D > 0$ and the other when $D < 0$.

(**b**) Now click on the reset button in the upper right corner of the screen. This will reset the value of the D to its initial setting of $D = 0$.

(**c**) Change the function to $g(t) = \cos(t)$ and repeat part (1) for the cosine function. Does changing the value of D affect the graph of $y = \cos(t) + D$ in the same way that changing the value for D affects the graph of $y = \sin(t) + D$?

2. Now change the value of A to 0.5, the value of B to 2, and the value of C to 0.5. The graph of $g(t) = \cos(t)$ will still be displayed but we will now have $f(t) = 0.5\cos(2(t - 0.5)) + D$. Does changing the value of D affect the graph of $y = 0.5\cos(2(t-0.5)) + D$ affect the sinusoidal wave in the same way that changing the value for D affects the graph of $y = \cos(t)$?

Part 2 – Using a Graphing Utility

Make sure your graphing utility is set to radian mode.

1. We will first examine the graph of $y = \cos(x) + D$ for four different values of D. Graph the five functions:

$y = \cos(x)$	$y = \cos(x) + 1$	$y = \cos(x) + 2$
	$y = \cos(x) - 1$	$y = \cos(x) - 2$

using the following settings for the viewing window: $0 \leq x \leq 2\pi$ and $-3 \leq y \leq 3$. Examine these graphs closely and describe the difference between the graph of $y = \cos(x) + D$ and the graph of $y = \cos(x)$ for these values of D.

2. Clear the graphics screen. We will now examine the graph of $y = 0.5\cos(2(x - 0.5)) + D$ for two different values of D. Graph the following three functions:

$$y = 0.5\cos(2(x - 0.5)) \qquad y = 0.5\cos(2(x - 0.5)) + 2$$
$$y = 0.5\cos(2(x - 0.5)) - 2$$

using the following settings for the viewing window: $0 \le x \le 2\pi$ and $-3 \le y \le 3$. Examine these graphs closely and describe the difference between the graph of $y = 0.5\cos(2(x - 0.5)) + D$ and the graph of $y = 0.5\cos(2(x - 0.5))$ for these values of D.

By exploring the graphs in Activity 2.12, we should notice that the graph of $y = A\sin(B(t - C)) + D$ is the graph of $y = A\sin(B(t - C))$ shifted up D units when $D > 0$ and shifted down $|D|$ units when $D < 0$. When working with a sinusoidal graph, such a vertical translation is called a **vertical shift**. This is illustrated in Figure 2.15 for a situation in which $D > 0$.

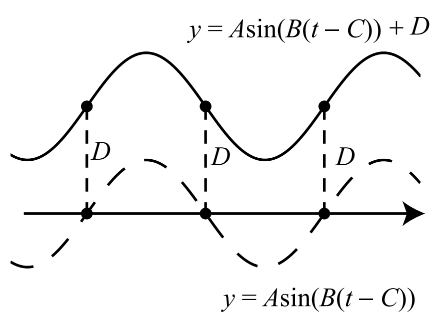


Figure 2.15: Graphs of $y = A\sin(B(t-C))$ (dashes) and $y = A\sin(B(t-C))+D$ (solid) for $D > 0$.

The y-axis is not shown in Figure 2.15 because this shows a general graph with a phase shift.

What we have done in Activity 2.12 is to start with a graph such as $y = A\sin(B(t-C))$ and added a constant to the dependent variable to get $y = A\sin(B(t-C)) + D$. So when t stays the same, we are adding D to the dependent variable. The effect is to translate the entire graph up by D units if $D > 0$ and down by $|D|$ units if $D < 0$.

Amplitude, Period, Phase Shift, and Vertical Shift

The following is a summary of the work we have done in this section dealing with amplitude, period, phase shift, and vertical shift for a sinusoidal function.

Let A, B, C, and D be nonzero real numbers with $B > 0$. For

$$y = A\sin(B(x-C)) + D \quad \text{and} \quad y = A\cos(B(x-C)+D):$$

1. **The amplitude** of the sinusoidal graph is $|A|$.
   - If $|A| > 1$, then there is a vertical stretch of the pure sinusoid by a factor of $|A|$.
   - If $|A| < 1$, then there is a vertical contraction of the pure sinusoid by a factor of $|A|$.
2. **The period** of the sinusoidal graph is $\dfrac{2\pi}{B}$.
   - When $B > 1$, there is a horizontal compression of the graphs.
   - When $0 < B < 1$, there is a horizontal stretch of the graph.
3. **The phase shift** of the sinusoidal graph is $|C|$.
   - If $C > 0$, there is a horizontal shift of C units to the right.
   - If $C < 0$, there is a horizontal shift of $|C|$ units to the left.
4. **The vertical shift** of the sinusoidal graph is $|D|$.
   - If $D > 0$, the vertical shift is D units up.
   - If $D < 0$, the vertical shift is $|D|$ units down.

Example 2.13 (The Graph of a Sinusoid)
This example will illustrate how to use the characteristics of a sinusoid and will serve as an introduction to the more general discussion that follows. The graph of $y = 3\sin\left(4\left(t - \frac{\pi}{8}\right)\right) + 2$ will look like the graph in Figure 2.16: Notice that the axes have not yet been drawn. We want to state the coordinates of the points P, Q, R, S, and T. There are several choices but we will make the point P as close to the origin as possible. Following are the important characteristics of this sinusoid:

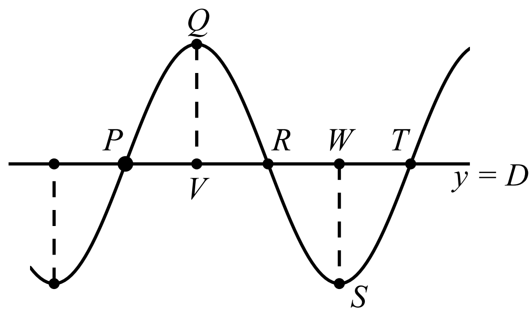


Figure 2.16: Graph of a Sinusoid

- The amplitude is 3.
- The period is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.
- The phase shift is $\dfrac{\pi}{8}$.
- The vertical shift is 2.

So for the graph in Figure 2.16, we can make the following conclusions.

- Since the vertical shift is 2, the horizontal line that is the center line of the sinusoid is $y = 2$.
- Since the phase shift is $\dfrac{\pi}{8}$ and this is a sine function, the coordinates of P are $\left(\dfrac{\pi}{8}, 2\right)$.
- Since the period is $\dfrac{\pi}{2}$, the t-coordinate of R is $\dfrac{\pi}{8} + \dfrac{1}{2}\left(\dfrac{\pi}{2}\right) = \dfrac{3\pi}{8}$. So the coordinates of R are $\left(\dfrac{3\pi}{8}, 2\right)$.
- Since the period is $\dfrac{\pi}{2}$, the t-coordinate of T is $\dfrac{\pi}{8} + \dfrac{\pi}{2} = \dfrac{5\pi}{8}$. So the coordinates of T are $\left(\dfrac{5\pi}{8}, 2\right)$.
- Since the period is $\dfrac{\pi}{2}$, the t-coordinate of Q is $\dfrac{\pi}{8} + \dfrac{1}{4}\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{4}$. Also, since the amplitude is 3, the y-coordinate of Q is $2 + 3 = 5$. So the coordinates of Q are $\left(\dfrac{\pi}{4}, 5\right)$.

- Since the period is $\dfrac{\pi}{2}$, the t-coordinate of S is $\dfrac{\pi}{8}+\dfrac{3}{4}\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}$. Also, since the amplitude is 3, the y-coordinate of S is $2-3=-1$. So the coordinates of S are $\left(\dfrac{\pi}{2},-1\right)$.

We can verify all of these results by using a graphing utility to draw the graph of $y=3\sin\left(4\left(t-\dfrac{\pi}{8}\right)\right)+2$ using $0\leq t\leq\dfrac{5\pi}{8}$ and $-2\leq y\leq 6$. If the utility allows, set the t-scale to one-quarter of a period, which is $\dfrac{\pi}{8}$.

Important Notes about Sinusoids

We can use $y=A\sin(B(t-C))+D$ or $y=A\cos(B(t-C))+D$ for a graph like the one in Figure 2.17.

- For both of these functions, the amplitude, the period, the vertical shift, and the horizontal shift will be the same.
- The difference between $y=A\sin(B(t-C))+D$ and $y=A\cos(B(t-C))+D$ will be the value of C for the phase shift.
- The horizontal line shown is not the t-axis. It is the horizontal line $y=D$, which we often call the **center line** for the sinusoid.

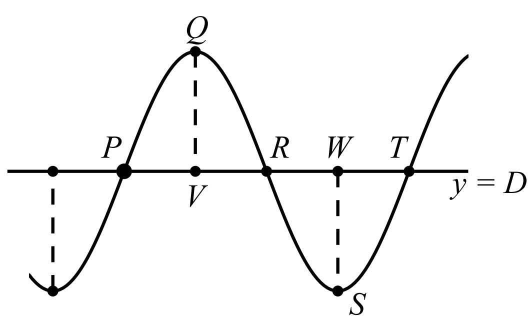


Figure 2.17: Graph of a Sinusoid

So to use the results about sinusoids and Figure 2.17, we have:

1. The amplitude, which we will call *amp*, is equal to the lengths of the segments QV and WS.

2. The period, which we will call *per*, is equal to $\dfrac{2\pi}{B}$. In addition, the lengths of the segments PV, VR, RW, and WT are equal to $\dfrac{1}{4}per$.

3. For $y = A\sin(B(x-C))+D$, we can use the point P for the phase shift. So the t-coordinate of the point P is C and P has coordinates (C, D). We can determine the coordinates of the other points as need by using the amplitude and period. For example:

   - The point Q has coordinates $\left(C + \dfrac{1}{4}per, D + amp\right)$.
   - The point R has coordinates $\left(C + \dfrac{1}{2}per, D\right)$.
   - The point S has coordinates $\left(C + \dfrac{3}{4}per, D - amp\right)$
   - The point T has coordinates $(C + per, D)$.

4. For $y = A\cos(B(x-C))+D$, we can use the point Q for the phase shift. So the t-coordinate of the point Q is C and Q has coordinates $(C, D + amp)$. We can determine the coordinates of the other points as need by using the amplitude and period. For example:

   - The point P has coordinates $\left(C - \dfrac{1}{4}per, D\right)$.
   - The point R has coordinates $\left(C + \dfrac{1}{4}per, D\right)$.
   - The point S has coordinates $\left(C + \dfrac{1}{2}per, D - amp\right)$
   - The point T has coordinates $\left(C + \dfrac{3}{4}per, D\right)$.

Please note that it is not necessary to try to remember all of the facts in items (3) and (4). What we should remember is how to use the concepts of one-quarter of a period and the amplitude illustrated in items (3) and (4). This will be done in the next two progress checks, which in reality are guided examples.

Progress Check 2.14 (Graphing a Sinusoid)
The characteristics of a sinusoid can be helpful in setting an appropriate viewing window when producing a usable graph of a sinusoid on a graphing utility. This

is especially true when the period is small or large. For example, consider the sinusoidal function

$$y = 6.3\cos(50\pi(t + 0.01)) + 2.$$

For this sinusoid:

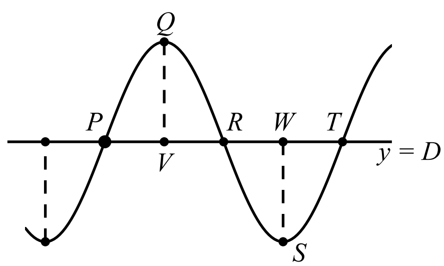
1. What is the amplitude?

2. What is the period?

3. What is the phase shift?

4. What is the vertical shift?

5. Use this information to determine coordinates for the point Q in the following diagram.



6. Now determine the coordinates of points P, R, S, and T.

7. Use this information and a graphing utility to draw a graph of (slightly more than) one period of this sinusoid that shows the points P, Q, and T.

Progress Check 2.15 (Determining an Equation for a Sinusoid)
We will determine two equations for the sinusoid shown in Figure 2.18.

1. Determine the coordinates of the points Q and R. The vertical distance between these two points is equal to two times the amplitude. Use the y-coordinates of these points to determine two times the amplitude and then the amplitude.

2. The center line for the sinusoid is half-way between the high point Q and the low point R. Use the y-coordinates of Q and R to determine the center line $y = D$. This will be the vertical shift.

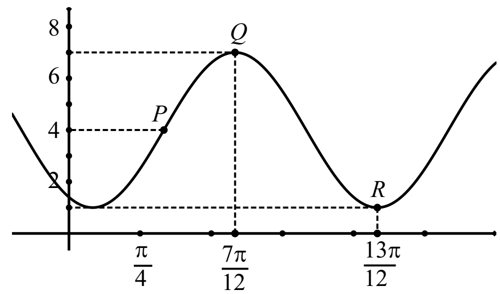


Figure 2.18: The Graph of a Sinusoid

3. The horizontal distance between Q and R is equal to one-half of a period. Use the t-coordinates of Q and R to determine the length of one-half of a period and hence, the period. Use this to determine the value of B.

4. We will now find an equation of the form $y = A\cos(B(t - C)) + D$. We still need the phase shift C. Use the point Q to determine the phase shift and hence, the value of C. We now have values for A, B, C, and D for the equation $y = A\cos(B(t - C)) + D$.

5. To determine an equation of the form $y = A\sin(B(t - C')) + D$, we will use the point P to determine the phase shift C'. (A different symbol was used because C' will be different than C in part (4).) Notice that the y-coordinate of P is 4 and so P lies on the center line. We can use the fact that the horizontal distance between P and Q is equal to one-quarter of a period. Determine the t coordinate of P, which will be equal to C'. Now write the equation $y = A\sin(B(t - C')) + D$ using the values of A, B, C', and D that we have determined.

We can check the equations we found in parts (4) and (5) by graphing these equations using a graphing utility.

Summary of Section 2.2

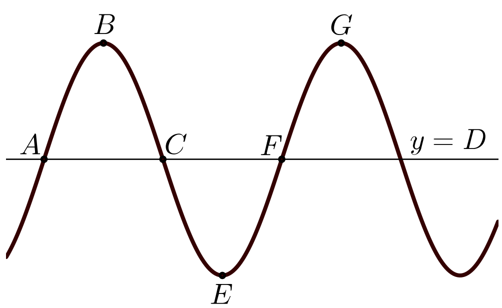
In this section, we studied the following important concepts and ideas: For a sinusoidal function of the form $f(t) = A\sin(B(t-C))+D$ or $f(t) = A\cos(B(t-C)) + D$ where A, B, C, and D are real numbers with $B > 0$:

- The value of $|A|$ is the amplitude of the sinusoidal function.

- The value of B determines the period of the sinusoidal function. the period is equal to $\dfrac{2\pi}{B}$.
- The value of C is the phase shift (horizontal shift) of the sinusoidal function. The graph is shifted to the right if $C > 0$ and shifted to the left if $C < 0$.
- The value of D is the vertical shift of the sinusoid. The horizontal line $y = D$ is the so-called **center line** for the graph of the sinusoidal function.
- The important notes about sinusoids starting on page 103.

Exercises for Section 2.2

1. The following is a graph of slightly more than one period of a sinusoidal function. Six points are labeled on the graph.



For each of the following sinusoidal functions:

- State the amplitude, period, phase shift, and vertical shift.
- State the coordinates of the points A, B, C, E, F, and G. Since the functions are periodic, there are several correct answers. For these functions, make the point A be as close to the origin as possible.

Notice that the horizontal line is not the horizontal axis but rather, the center line $y = D$ for the sinusoid.

⋆ **(a)** $y = 2\sin(\pi x)$

(b) $y = 7.2\cos(2x)$

⋆ **(c)** $y = 3\sin\left(x - \dfrac{\pi}{4}\right)$

(d) $y = 3\sin\left(x + \dfrac{\pi}{4}\right)$

(e) $y = 4\cos\left(x - \dfrac{\pi}{3}\right)$

(f) $y = 2.8\cos\left(2\left(x - \dfrac{\pi}{3}\right)\right)$

⋆ **(g)** $y = 4\sin\left(2\left(x - \dfrac{\pi}{4}\right)\right) + 1$

(h) $y = -4\cos\left(2\left(x + \dfrac{\pi}{4}\right)\right) + 1$

(i) $y = 3\cos\left(2\pi x - \dfrac{\pi}{2}\right)$

(j) $y = -1.75\sin\left(2x - \dfrac{\pi}{3}\right) + 2$

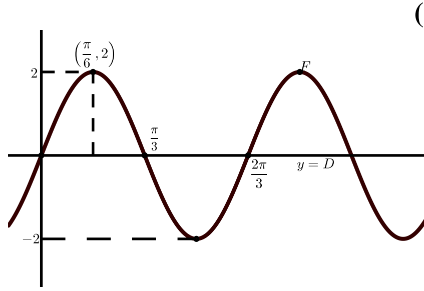
(k) $y = 5\sin(120\pi x)$

(l) $y = 40\sin\left(50\pi\left(x - \dfrac{1}{100}\right)\right)$

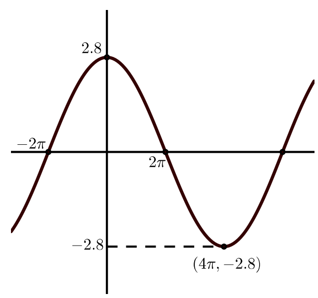
2. Each of the following graphs is a graph of a sinusoidal function. In each case:

- Determine the amplitude of the sinusoidal function.
- Determine the period of the sinusoidal function.
- Determine the vertical shift of the sinusoidal function.
- Determine an equation of the form $y = A\sin(B(x - C)) + D$ that produces the given graph.
- Determine an equation of the form $y = A\cos(B(x - C)) + D$ that produces the given graph.

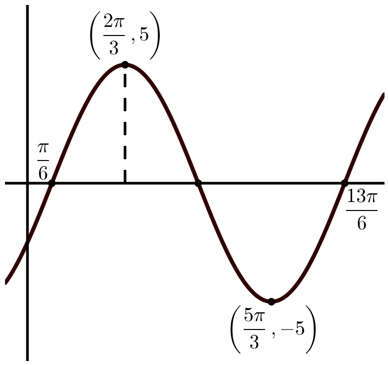
⋆ **(a)**



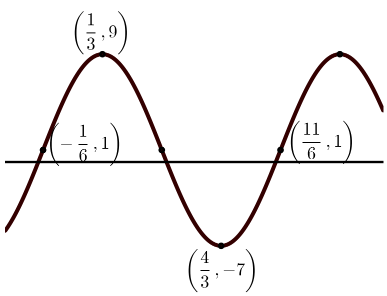
(b)



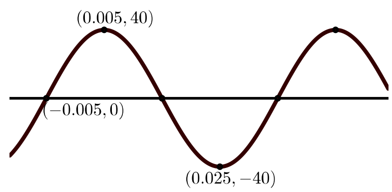
(c)



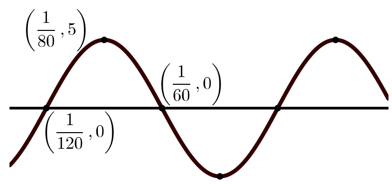
⋆ **(d)**



(e)



(f)



3. Each of the following web links is to an applet on Geogebratube. For each one, the graph of a sinusoidal function is given. The goal is to determine a function of the form

$$f(x) = A\sin(B(x - C)) + D \quad \text{or} \quad f(x) = A\cos(B(x - C)) + D$$

as directed in the applet. There are boxes that must be used to enter the values of A, B, C, and D.

(a) `http://gvsu.edu/s/09f`
(b) `http://gvsu.edu/s/09g`
(c) `http://gvsu.edu/s/09h`
(d) `http://gvsu.edu/s/09i`
(e) `http://gvsu.edu/s/09j`
(f) `http://gvsu.edu/s/09k`

2.3 Applications and Modeling with Sinusoidal Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

Let A, B, C, and D be real number constants with $B > 0$ and consider the graph of $f(t) = A\sin(B(t - C)) + D$ or $f(t) = A\cos(B(t - C)) + D$.

- What does frequency mean?
- How do we model periodic data accurately with a sinusoidal function?
- What is a mathematical model?
- Why is it reasonable to use a sinusoidal function to model periodic phenomena?

In Section 2.2, we used the diagram in Figure 2.19 to help remember important facts about sinusoidal functions.

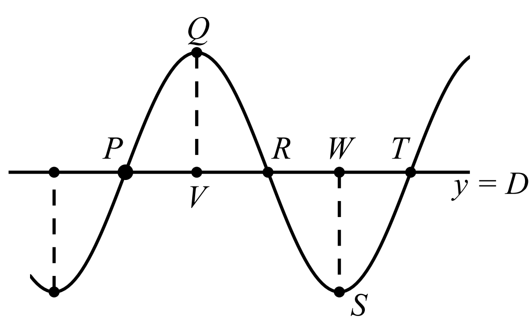


Figure 2.19: Graph of a Sinusoid

For example:

- The horizontal distance between a point where a maximum occurs and the next point where a minimum occurs (such as points Q and S) is one-half of a period. This is the length of the segment from V to W in Figure 2.19.

- The vertical distance between a point where a minimum occurs (such as point S) and a point where is maximum occurs (such as point Q) is equal to two times the amplitude.
- The center line $y = D$ for the sinusoid is half-way between the maximum value at point Q and the minimum value at point S. The value of D can be found by calculating the average of the y-coordinates of these two points.
- The horizontal distance between any two successive points on the line $y = D$ in Figure 2.19 is one-quarter of a period.

In Progress Check 2.16, we will use some of these facts to help determine an equation that will model the volume of blood in a person's heart as a function of time. A **mathematical model** is a function that describes some phenomenon. For objects that exhibit periodic behavior, a sinusoidal function can be used as a model since these functions are periodic. However, the concept of frequency is used in some applications of periodic phenomena instead of the period.

Definition. The **frequency** of a sinusoidal function is the number of periods (or cycles) per unit time.

A typical unit for frequency is the **hertz**. One hertz (Hz) is one cycle per second. This unit is named after Heinrich Hertz (1857 – 1894).

Since frequency is the number of cycles per unit time, and the period is the amount of time to complete one cycle, we see that frequency and period are related as follows:

$$\text{frequency} = \frac{1}{\text{period}}.$$

Progress Check 2.16 (Volume of Blood in a Person's Heart)
The volume of the average heart is 140 milliliters (ml), and it pushes out about one-half its volume (70 ml) with each beat. In addition, the *frequency* for the heartbeat of a well-trained athlete heartbeat for a well-trained athlete is 50 beats (cycles) per minute. We will model the volume, $V(t)$ (in milliliters) of blood in the heart as a function of time t measured in seconds. We will use a sinusoidal function of the form

$$V(t) = A\cos(B(t - C)) + D.$$

If we choose time 0 minutes to be a time when the volume of blood in the heart is the maximum (the heart is full of blood), then it reasonable to use a cosine function for our model since the cosine function reaches a maximum value when its input is 0 and so we can use $C = 0$, which corresponds to a phase shift of 0. So our function can be written as $V(t) = A\cos(Bt) + D$.

1. What is the maximum value of $V(t)$? What is the minimum value of $V(t)$? Use these values to determine the values of A and D for our model? Explain.

2. Since the frequency of heart beats is 50 beats per minute, we know that the time for one heartbeat will be $\frac{1}{50}$ of a minute. Determine the time (in seconds) it takes to complete one heartbeat (cycle). This is the period for this sinusoidal function. Use this period to determine the value of B. Write the formula for $V(t)$ using the values of A, B, C, and D that have been determined.

Example 2.17 (Continuation of Progress Check 2.16)
Now that we have determined that

$$V(t) = 35\cos\left(\frac{5\pi}{3}t\right) + 105$$

(where t is measured in seconds since the heart was full and $V(t)$ is measured in milliliters) is a model for the amount of blood in the heart, we can use this model to determine other values associated with the amount of blood in the heart. For example:

- We can determine the amount of blood in the heart 1 second after the heart was full by using $t = 1$.

$$V(1) = 35\cos\left(\frac{5\pi}{3}\right) + 105 \approx 122.5.$$

  So we can say that 1 second after the heart is full, there will be 122.5 milliliters of blood in the heart.

- In a similar manner, 4 seconds after the heart is full of blood, there will be 87.5 milliliters of blood in the heart since

$$V(4) = 35\cos\left(\frac{20\pi}{3}\right) + 105 \approx 87.5.$$

- Suppose that we want to know at what times after the heart is full that there will be 100 milliliters of blood in the heart. We can determine this if we can solve the equation $V(t) = 100$ for t. That is, we need to solve the equation

$$35\cos\left(\frac{5\pi}{3}t\right) + 105 = 100.$$

Although we will learn other methods for solving this type of equation later in the book, we can use a graphing utility to determine approximate solutions for this equation. Figure 2.20 shows the graphs of $y = V(t)$ and $y = 100$. To solve the equation, we can use a graphing utility that allows us to determine or approximate the points of intersection of two graphs. (This can be done using most Texas Instruments calculators and Geogebra.) The idea is to find the coordinates of the points P, Q, and R in Figure 2.20.

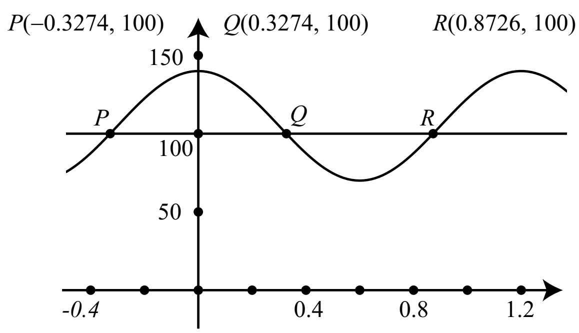


Figure 2.20: Graph of $V(t) = 35\cos\left(\frac{5\pi}{3}t\right) + 105$ and $y = 100$

We really only need to find the coordinates of one of those points since we can use properties of sinusoids to find the others. For example, we can determine that the coordinates of P are $(-0.3274, 100)$. Then using the fact that the graph of $y = V(t)$ is symmetric about the y-axis, we know the coordinates of Q are $(0.3274, 100)$. We can then use the periodic property of the function, to determine the t-coordinate of R by adding one period to the t-coordinate of P. This gives $-0.3274 + \frac{6}{5} = 0.8726$, and the coordinates of R are $(0.8726, 100)$. We can also use the periodic property to determine as many solutions of the equation $V(t) = 100$ as we like.

Progress Check 2.18 (Hours of Daylight)
The summer solstice in 2014 was on June 21 and the winter solstice was on December 21. The maximum hours of daylight occurs on the summer solstice and the minimum hours of daylight occurs on the winter solstice. According to the U.S. Naval Observatory website,

http://aa.usno.navy.mil/data/docs/Dur_OneYear.php,

the number of hours of daylight in Grand Rapids, Michigan on June 21, 2014 was 15.35 hours, and the number of hours of daylight on December 21, 2014 was 9.02 hours. This means that in Grand Rapids,

- The maximum number of hours of daylight was 15.35 hours and occurred on day 172 of the year.
- The minimum number of hours of daylight was 9.02 hours and occurred on day 355 of the year.

1. Let y be the number of hours of daylight in 2014 in Grand Rapids and let t be the day of the year. Determine a sinusoidal model for the number of hours of daylight y in 2014 in Grand Rapids as a function of t.

2. According to this model,

(**a**) How many hours of daylight were there on March 10, 2014?

(**b**) On what days of the year were there 13 hours of daylight?

Determining a Sinusoid from Data

In Progress Check 2.18 the values and times for the maximum and minimum hours of daylight. Even if we know some phenomenon is periodic, we may not know the values of the maximum and minimum. For example, the following table shows the number of daylight hours (rounded to the nearest hundredth of an hour) on the first of the month for Edinburgh, Scotland ($55°57'$ N, $3°12'$ W).

We will use a sinusoidal function of the form $y = A\sin(B(t - C)) + D$, where y is the number of hours of daylight and t is the time measured in months to model this data. We will use 1 for Jan., 2 for Feb., etc. As a first attempt, we will use 17.48 for the maximum hours of daylight and 7.08 for the minimum hours of daylight.

- Since $17.48 - 7.08 = 10.4$, we see that the amplitude is 5.2 and so $A = 5.2$.

Jan	Feb	Mar	Apr	May	June
7.08	8.60	10.73	13.15	15.40	17.22

July	Aug	Sept	Oct	Nov	Dec
17.48	16.03	13.82	11.52	9.18	7.40

Table 2.2: Hours of Daylight in Edinburgh

- The vertical shift will be $7.08 + 5.2 = 12.28$ and so $D = 12.28$.
- The period is 12 months and so $B = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$.
- The maximum occurs at $t = 7$. For a sine function, the maximum is one-quarter of a period from the time when the graph crosses its horizontal axis. The period is 12 months, and so we get a phase shift of 4 to the right and $C = 4$.

So we will use the function $y = 5.2 \sin\left(\dfrac{\pi}{6}(t-4)\right) + 12.28$ to model the number of hours of daylight. Figure 2.21 shows a scatterplot for the data and a graph of this function. Although the graph fits the data reasonably well, it seems we should

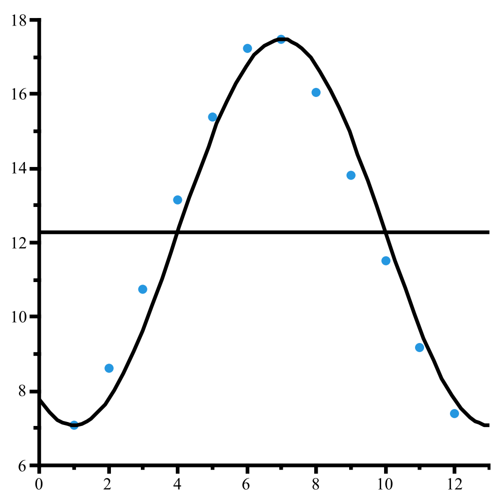


Figure 2.21: Hours of Daylight in Edinburgh

be able to find a better model. One of the problems is that the maximum number of hours of daylight does not occur on July 1. It probably occurs about 10 days earlier. The minimum also does not occur on January 1 and is probably somewhat less that 7.08 hours. So we will try a maximum of 17.50 hours and a minimum of 7.06 hours. Also, instead of having the maximum occur at $t = 7$, we will say it occurs at $t = 6.7$. Using these values, we have $A = 5.22$, $B = \dfrac{\pi}{6}$, $C = 3.7$, and $D = 12.28$. Figure 2.22 shows a scatterplot of the data and a graph of

$$y = 5.22\sin\left(\frac{\pi}{6}(t - 3.7)\right) + 12.28. \tag{1}$$

This appears to model the data very well. One important thing to note is that

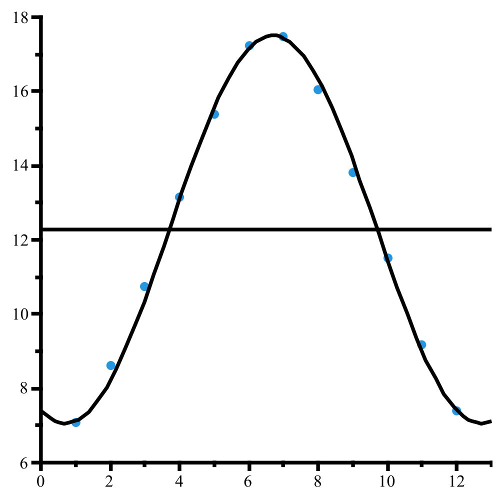


Figure 2.22: Hours of Daylight in Edinburgh

when trying to determine a sinusoid that "fits" or models actual data, there is no single correct answer. We often have to find one model and then use our judgment in order to determine a better model. There is a mathematical "best fit" equation for a sinusoid that is called the **sine regression equation**. Please note that we need to use some graphing utility or software in order to obtain a sine regression equation. Many Texas Instruments calculators have such a feature as does the software Geogebra. Following is a sine regression equation for the number of hours of daylight in Edinburgh shown in Table 2.2 obtained from Geogebra.

$$y = 5.153\sin(0.511t - 1.829) + 12.174. \tag{2}$$

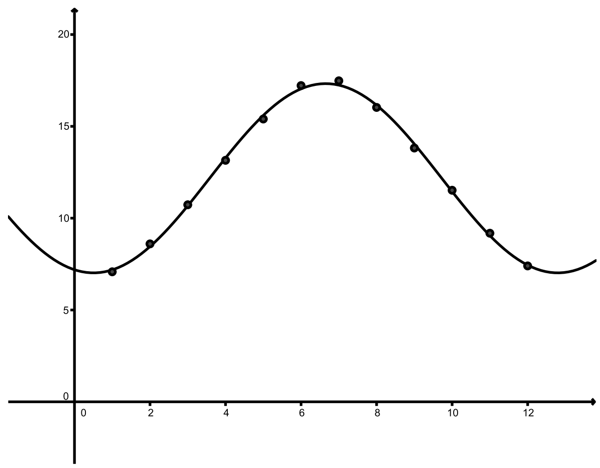


Figure 2.23: Hours of Daylight in Edinburgh

A scatterplot with a graph of this function is shown in Figure 2.23.

Note: See the supplements at the end of this section for instructions on how to use Geogebra and a Texas Instruments TI-84 to determine a sine regression equation for a given set of data.

It is interesting to compare equation (1) and equation (2), both of which are models for the data in Table 2.2. We can see that both equations have a similar amplitude and similar vertical shift, but notice that equation (2) is not in our standard form for the equation of a sinusoid. So we cannot immediately tell what that equation is saying about the period and the phase shift. In this next activity, we will learn how to determine the period and phase shift for sinusoids whose equations are of the form $y = a\sin(bt + c) + d$ or $y = a\cos(bt + c) + d$.

Activity 2.19 (Working with Sinusoids that Are Not In Standard Form)
So far, we have been working with sinusoids whose equations are of the form $y = A\sin(B(t - C)) + D$ or $y = A\cos(B(t - C)) + D$. When written in this form, we can use the values of A, B, C, and D to determine the amplitude, period, phase shift, and vertical shift of the sinusoid. We must always remember, however, that to do this, the equation must be written in exactly this form. If we have an equation in a slightly different form, we have to determine if there is a way to use algebra to rewrite the equation in the form $y = A\sin(B(t - C)) + D$ or

$y = A\cos(B(t - C)) + D$. Consider the equation

$$y = 2\sin\left(3t + \frac{\pi}{2}\right).$$

1. Use a graphing utility to draw the graph of this equation with $-\frac{\pi}{3} \le t \le \frac{2\pi}{3}$ and $-3 \le y \le 3$. Does this seem to be the graph of a sinusoid? If so, can you use the graph to find its amplitude, period, phase shift, and vertical shift?

2. It is possible to verify any observations that were made by using a little algebra to write this equation in the form $y = A\sin(B(t - C)) + D$. The idea is to rewrite the argument of the sine function, which is $3t + \frac{\pi}{2}$ by "factoring a 3" from both terms. This may seem a bit strange since we are not used to using fractions when we factor. For example, it is quite easy to factor $3y + 12$ as

$$3y + 12 = 3(y + 4).$$

In order to "factor" three from $\frac{\pi}{2}$, we basically use the fact that $3 \cdot \frac{1}{3} = 1$. So we can write

$$\begin{aligned}\frac{\pi}{2} &= 3 \cdot \frac{1}{3} \cdot \frac{\pi}{2} \\ &= 3 \cdot \frac{\pi}{6}\end{aligned}$$

Now rewrite $3t + \frac{\pi}{2}$ by factoring a 3 and then rewrite $y = 2\sin\left(3t + \frac{\pi}{2}\right)$ in the form $y = A\sin(B(t - C)) + D$.

3. What is the amplitude, period, phase shift, and vertical shift for $y = 2\sin\left(3t + \frac{\pi}{2}\right)$?

In Activity 2.19, we did a little factoring to show that

$$\begin{aligned}y &= 2\sin\left(3t + \frac{\pi}{2}\right) = 2\sin\left(3\left(t + \frac{\pi}{6}\right)\right) \\ y &= 2\sin\left(3\left(t - \left(-\frac{\pi}{6}\right)\right)\right)\end{aligned}$$

So we can see that we have a sinusoidal function and that the amplitude is 3, the period is $\frac{2\pi}{3}$, the phase shift is $-\frac{\pi}{6}$, and the vertical shift is 0.

In general, we can see that if b and c are real numbers, then

$$bt + c = b\left(t + \frac{c}{b}\right) = b\left(t - \left(-\frac{c}{b}\right)\right).$$

This means that

$$y = a\sin(bt + c) + d = a\sin\left(b\left(t - \left(-\frac{c}{b}\right)\right)\right) + d.$$

So we have the following result:

If $y = a\sin(bt + c) + d$ or $y = a\cos(bt + c) + d$, then

- The amplitude of the sinusoid is $|a|$.
- The period of the sinusoid is $\dfrac{2\pi}{b}$.
- The phase shift of the sinusoid is $-\dfrac{c}{b}$.
- The vertical shift of the sinusoid is d.

Progress Check 2.20 (The Other Form of a Sinusoid)

1. Determine the amplitude, period, phase shift, and vertical shift for each of the following sinusoids. Then use this information to graph one complete period of the sinusoid and state the coordinates of a high point, a low point, and a point where the sinusoid crosses the center line.

   (a) $y = -2.5\cos\left(3x + \dfrac{\pi}{3}\right) + 2.$

   (b) $y = 4\sin\left(100\pi x - \dfrac{\pi}{4}\right)$

2. We determined two sinusoidal models for the number of hours of daylight in Edinburgh, Scotland shown in Table 2.2. These were

$$y = 5.22\sin\left(\frac{\pi}{6}(t - 3.7)\right) + 12.28$$
$$y = 5.153\sin(0.511t - 1.829) + 12.174$$

   The second equation was determined using a sine regression feature on a graphing utility. Compare the amplitudes, periods, phase shifts, and vertical shifts of these two sinusoidal functions.

Supplement – Sine Regression Using Geogebra

Before giving written instructions for creating a sine regression equation in Geogebra, it should be noted that there is a Geogebra Playlist on the Grand Valley State University Math Channel on YouTube. The web address is

http://gvsu.edu/s/QA

The video screencasts that are of most interest for now are:

- Geogebra – Basic Graphing
- Geogebra – Copying the Graphics View
- Geogebra – Plotting Points
- Geogebra – Sine Regression

To illustrate the procedure for a sine regression equation using Geogebra, we will use the data in Table 2.2 on page 115.

Step 1. Set a viewing window that is appropriate for the data that will be used.

Step 2. Enter the data points. There are three ways to do this.

- Perhaps the most efficient way to enter points is to use the spreadsheet view. To do this, click on the View Menu and select Spreadsheet. A small spreadsheet will open on the right. Although you can use any sets of rows and columns, an easy way is to use cells A1 and B1 for the first data point, cells A2 and B2 for the second data point, and so on. So the first few rows in the spreadsheet would be:

	A	B
1	1	7.08
2	2	8.6
3	3	10.73

  Once all the data is entered, to plot the points, select the rows and columns in the spreadsheet that contain the data, then click on the small downward arrow on the bottom right of the button with the label $\{1, 2\}$ and select "Create List of Points." A small pop-up screen will appear in which the list of points can be given a name. The default name is "l1" but that can be changed if desired. Now click on the Create button in the lower right side of the pop-up screen. If a proper viewing window has been set, the points should appear in the graphics view. Finally, close the spreadsheet view.

- Enter each point separately as an order pair. For example, for the first point in Table 2.2, we would enter $(1, 7.08)$. In this case, each point will be given a name such as A, B, C, etc.
- Enter all the points in a list. For example (for a smaller set of points), we could enter something like

$$pts = \{(-3,3), (-2,-1), (0,1), (1,3), (3,0)\}$$

Notice that the list of ordered pairs must be enclosed in braces.

Step 3. Use the FitSin command. How this is used depends on which option was used to enter and plot the data points.

- If a list of points has been created (such as one named list1), simply enter

$$f(x) = \text{FitSin[list1]}$$

All that is needed is the name of the list inside the brackets.

- If separate data points have been enter, include the names of all the points inside the brackets and separate them with commas. An abbreviated version of this is

$$f(x) = \text{FitSin[A, B, C]}$$

The sine regression equation will now be shown in the Algebra view and will be graphed in the graphics view.

Step 4. Select the rounding option to be used. (This step could be performed at any time.) To do this, click on the Options menu and select Rounding.

Summary of Section 2.3

In this section, we studied the following important concepts and ideas:

- The **frequency** of a sinusoidal function is the number of periods (or cycles) per unit time.

$$\text{frequency} = \frac{1}{\text{period}}.$$

- A **mathematical model** is a function that describes some phenomenon. For objects that exhibit periodic behavior, a sinusoidal function can be used as a model since these functions are periodic.

- To determine a sinusoidal function that models a periodic phenomena, we need to determine the amplitude, the period, and the vertical shift for the periodic phenomena. In addition, we need to determine whether to use a cosine function or a sine function and the resulting phase shift.
- A **sine regression equation** can be determined that is a mathematical "best fit" for data from a periodic phenomena.

Exercises for Section 2.3

1. Determine the amplitude, period, phase shift, and vertical shift for each of the following sinusoids. Then use this information to graph one complete period of the sinusoid and state coordinates of a high point, a low point, and a point where the sinusoid crosses the center line.

⋆ **(a)** $y = 4\sin\left(\pi x - \dfrac{\pi}{8}\right)$.

⋆ **(b)** $y = 5\cos\left(4x + \dfrac{\pi}{2}\right) + 2$.

(c) $y = -3.2\cos\left(50\pi x - \dfrac{\pi}{2}\right)$.

(d) $y = 4.8\sin\left(\dfrac{1}{4}x + \dfrac{\pi}{8}\right)$.

2. Modeling a Heartbeat. For a given person at rest, suppose the heart pumps blood at a regular rate of about 75 pulses per minute. Also, suppose that the volume of this person's heart is approximately 150 milliliters (ml), and it pushes out about 54% its volume with each beat. We will model the volume, $V(t)$ of blood (in milliliters) in the heart at any time t, as a sinusoidal function of the form

$$V(t) = A\cos(Bt) + D.$$

(a) If we choose time 0 to be a time when the heart is full of blood, why is it reasonable to use a cosine function for our model?

⋆ **(b)** What is the maximum value of $V(t)$? What is the minimum value of $V(t)$? What does this tell us about the values of A and D? Explain.

⋆ **(c)** The *frequency* of a simple harmonic motion is the number of periods per unit time, or the number of pulses per minute in this example. How is the frequency f related to the period? What should be the value of B? Explain.

(d) Draw a graph (without a calculator) of your $V(t)$ using your values of A, B, and D, of two periods beginning at $t = 0$.

(e) Clearly identify the maximum and minimum values of $V(t)$ on the graph. What do these numbers tell us about the heart at these times?

3. The electricity supplied to residential houses is called alternating current (AC) because the current varies sinusoidally with time. The voltage which causes the current to flow also varies sinusoidally with time. In an alternating (AC) current circuit, the voltage V (in volts) as a function of time is a sinusoidal function of the form

$$V = V_0 \sin(2\pi f t), \tag{1}$$

where V_0 is a positive constant and f is the **frequency**. The frequency is the number of complete oscillations (cycles) per second. In the United States, f is 60 hertz (Hz), which means that the frequency is 60 cycles per second.

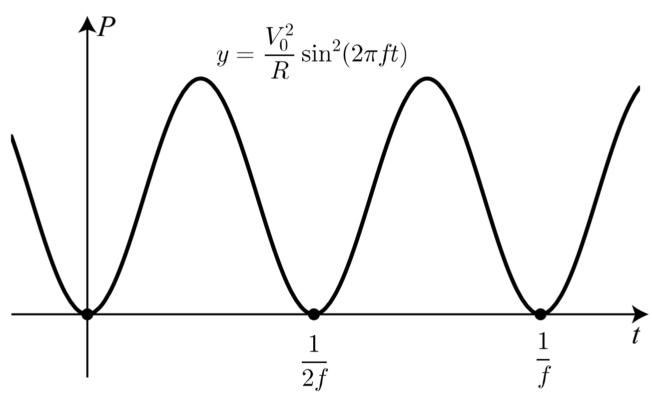
(a) What is the amplitude and what is the period of the sinusoidal function in (1)?

The **power** (in watts) delivered to a resistance R (in ohms) at any time t is given by

$$P = \frac{V^2}{R}. \tag{2}$$

(b) Show that $P = \dfrac{V_0^2}{R} \sin^2(2\pi f t)$.

(c) The graph of P as a function of time is shown below.



Assuming that this shows that P is a sinusoidal function of t, write P as a sinusoidal function of time t by using the negative of a cosine function with no phase shift.

(d) So we know that $P = \dfrac{V_0^2}{R}\sin^2(2\pi f t)$ and that P is equal to the sinusoidal function in part (c). Set the two expressions for P equal to each other and use the resulting equation to conclude that

$$\sin^2(2\pi f t) = \frac{1}{2}[1 - \cos(4\pi f t)].$$

4. The electricity supplied to residential houses is called alternating current (AC) because the current varies sinusoidally with time. The voltage which causes the current to flow also varies sinusoidally with time. Both current and voltage have a frequency of 60 cycles per second, but they have different phase shifts. (Note: A frequency of 60 cycles per second corresponds to a period of $\dfrac{1}{60}$ of a second.)

Let C be the current (in amperes), let V be the voltage (in volts), and let t be time (in seconds). The following list gives information that is known about C and V.

- The current C is a sinusoidal function of time with a frequency of 60 cycles per second, and it reaches its maximum of 5 amperes when $t = 0$ seconds.
- The voltage V is a sinusoidal function of time with a frequency of 60 cycles per second. As shown in the graphs on the next page, V "leads" the current in the sense that it reaches its maximum before the current reaches its maximum. ("Leading" corresponds to a negative phase shift, and "lagging" corresponds to a positive phase shift.) In this case, the voltage V leads the current by 0.003 seconds, meaning that it reaches its maximum 0.003 seconds before the current reaches its maximum.
- The peak voltage is 180 volts.
- There is no vertical shift on either the current or the voltage graph.

(a) Determine sinusoidal functions for both C and V.

(b) What is the voltage when the current is a maximum?

(c) What is the current when the voltage is a minimum?

(d) What is the current when the voltage is equal to zero?

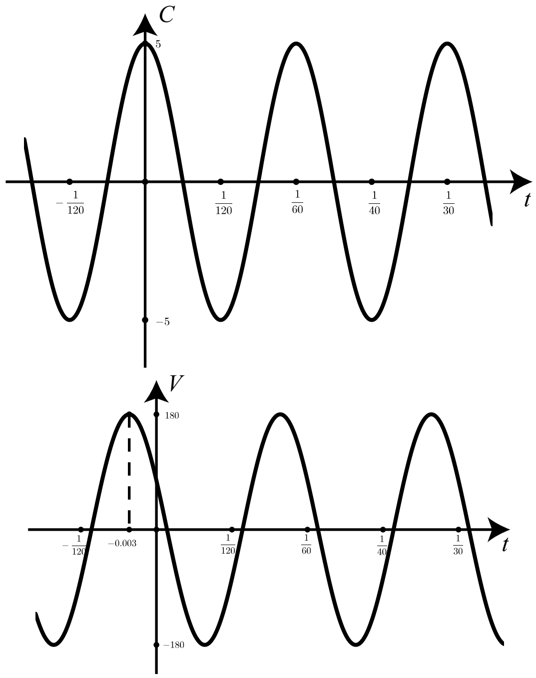


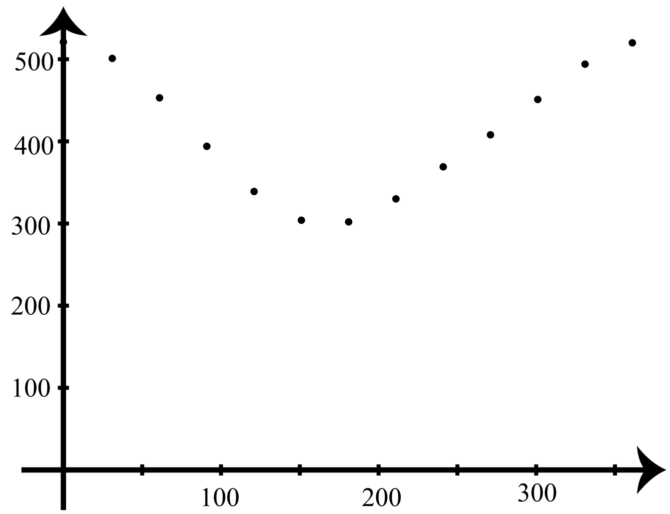
Figure 2.24: Current C and Voltage V As Functions of Time

5. We will let t be the number of the day of the year. The following table shows sunrise times (in minutes since midnight) for certain days of the year at Houghton, Michigan.

day	1	31	61	91	121	151	181
time	521	501	453	394	339	304	302

day	211	241	271	301	331	361	
time	330	369	408	451	494	520	

The points for this table are plotted on the following graph.



(**a**) Let t be the number of the day of the year and let y be the sunrise time in minutes since midnight at Houghton, MI. Determine a sinusoidal model for y as a function of t.

(**b**) To check the work in Part (a), use a graphing utility or Geogebra to plot the points in the table and superimpose the graph of the function from Part (a).

(**c**) Use Geogebra to determine a sinusoidal model for y as a function of t. This model will be in the form $y = a\sin(bt + c) + d$, where a, b, c, and d are real numbers.

(**d**) Determine the amplitude, period, phase shift, and vertical shift for the sinusoidal model in Part (c).

6. Modeling the Distance from the Earth to the Sun. The Earth's orbit around the sun is not a perfect circle. In 1609 Johannes Kepler published two of his famous laws of planetary motion, one of which states that planetary orbits are actually ellipses. So the distance from the Earth to the sun is not a constant, but varies over the course of its orbit (we will assume a 365 day year). According to the 1996 US Ephemeris[1], the distances from the sun to the Earth on the 21st of each month are given in Table 2.3. The distances are measured in Astronomical Units (AU), where 1 AU is approximately 149,597,900 kilometers.

[1] http://image.gsfc.nasa.gov/poetry/venus/q638.html

Month	Day of the year	Distance
January	21	0.9840
February	52	0.9888
March	80	0.9962
April	111	1.0050
May	141	1.0122
June	172	1.0163
July	202	1.0161
August	233	1.0116
September	264	1.0039
October	294	0.9954
November	325	0.9878
December	355	0.9837

Table 2.3: Distances from the Earth to the sun on the 21st of each month

A plot of this data with the day of the year along the horizontal axis and the distance from the Earth to the sun on the vertical axis is given in Figure 2.25.

We will use a sinusoidal function to model this data. That is, we will let $f(t)$ be the distance from the Earth to the Sun on day t of the year and that

$$f(t) = A\sin(B(t - C) + D.$$

(a) What are the maximum and minimum distances from the Earth to the sun given by the data? What does this tell us about the amplitude of $f(t)$? Use this to approximate the values of A and D in the model function f? What is the center line for this sinusoidal model?

(b) The period of this sinusoidal function is 365 days. What is the value of B for this sinusoidal function?

(c) Draw the center line you found in part (a) on the plot of the data in Figure 2.25. At approximately what value of t will the graph of f intersect this center line? How is this number related to the phase shift of the data? What is the value of C for this sinusoidal function?

(d) Use Geogebra to plot the points from the data in Table 2.3 and then use Geogebra to draw the graph of the sinusoidal model $f(t) = A\sin(B(t-C) + D$. Does this function model the data reasonably well?

(e) Use the sinusoidal model $f(t) = A\sin(B(t - C) + D$ to estimate the distance from the Earth to the Sun on July 4.

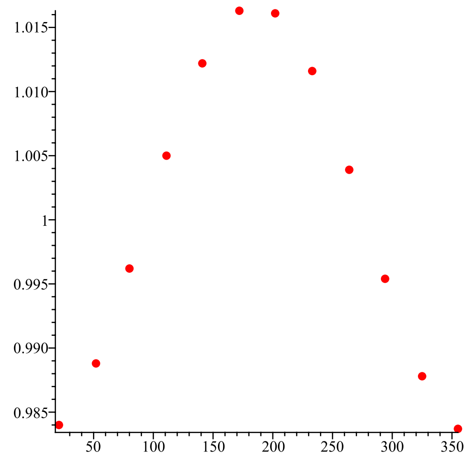


Figure 2.25: Distance from the Earth to the sun as a function of the day of the year

7. **Continuation of Exercise (6)**. Use Geogebra to plot the points from the data in Table 2.3. Then use the "FitSin" command in Geogebra to find a sinusoidal model for this data of the form $g(t) = a\sin(bt + c) + d$.

   What is the amplitude of this sinusoidal model? What is the period? What is the horizontal shift? What is the phase shift?

   How do these values compare with the corresponding values for the sinusoid $f(t) = A\sin(B(t - C)) + D$ obtained in Exercise (6)?

8. As the moon orbits the earth, the appearance of the moon changes. We see various lunar disks at different times of the month. These changes reappear during each lunar month. However, a lunar month is not exactly the same as the twelve months we use in our calendar today. A lunar month is the number of days it takes the moon to go through one complete cycle from a full moon (100% illumination) to the next full moon.

   The following data were gathered from the web site for the U.S. Naval Observatory. The data are geocentric values of the percent of the moon that is illuminated. That is, the percent of illumination is computed for a fictitious observer located at the center of the Earth.

Date	Percent Illumintated
3/1/2013	87%
3/3/2013	69%
3/5/2017	47%
3/7/2017	25%
3/9/2017	9%
3/12/2013	0%
3/13/2013	2%
3/15/2017	12%

Date	Percent Illumintated
3/17/2013	27%
3/19/2013	45%
3/21/2013	64%
3/23/2013	81%
3/25/2013	94%
3/27/2013	100%
3/29/2013	96%

(a) Determine a sinusoidal function of the form $y = A\cos(B(x-C)) + D$ to model this data. For this function, let x be the number of days since the beginning of March 2013 and let y be the percent of the moon that is illuminated. What is the amplitude, period, phase shift, and vertical shift of this sinusoidal function?

(b) Use Geogebra to draw a scatterplot of this data and superimpose the graph of the function from part (a).

(c) Use Geogebra to determine a sinusoidal function of the form $y = A\sin(Bx + K) + D$ to model this data and superimpose its graph on the scatterplot. What is the amplitude, period, phase shift, and vertical shift of this sinusoidal function?

9. Each of the following web links is to an applet on Geogebratube. For each one, data is plotted and in some cases, the actual data is shown in a spreadsheet on the right. The goal is to determine a function of the form

$$f(x) = A\sin(B(x-C)) + D \quad \text{or} \quad f(x) = A\cos(B(x-C)) + D$$

that fits the data as closely as possible. Each applet will state which type of function to use. There are boxes that must be used to enter the values of A, B, C, and D.

(a) http://gvsu.edu/s/09l

(b) http://gvsu.edu/s/09m

(c) http://gvsu.edu/s/09n

(d) http://gvsu.edu/s/09o

2.4 Graphs of the Other Trigonometric Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the important properties of the graph of $y = \tan(t)$. That is, what is the domain and what is the range of the tangent function, and what happens to the values of the tangent function at the points that are near points not in the domain of the tangent function?
- What are the important properties of the graph of $y = \sec(t)$. That is, what is the domain and what is the range of the secant function, and what happens to the values of the secant function at the points that are near points not in the domain of the secant function?
- What are the important properties of the graph of $y = \cot(t)$. That is, what is the domain and what is the range of the cotangent function, and what happens to the values of the cotangent function at the points that are near points not in the domain of the cotangent function?
- What are the important properties of the graph of $y = \csc(t)$. That is, what is the domain and what is the range of the cosecant function, and what happens to the values of the cosecant function at the points that are near points not in the domain of the cosecant function?

We have seen how the graphs of the cosine and sine functions are determined by the definition of these functions. We also investigated the effects of the constants A, B, C, and D on the graph of $y = A\sin(B(x - C)) + D$ and the graph of $y = A\cos(B(x - C)) + D$.

In the following beginning activity, we will explore the graph of the tangent function. Later in this section, we will discuss the graph of the secant function, and the graphs of the cotangent and cosecant functions will be explored in the exercises. One of the key features of these graphs is the fact that they all have *vertical asymptotes*. Important information about all four functions is summarized at the end of this section.

Beginning Activity

1. Use a graphing utility to draw the graph of $f(x) = \dfrac{1}{(x+1)(x-1)}$ using $-2 \le x \le 2$ and $-10 \le y \le 10$. If possible, use the graphing utility to draw the graphs of the vertical lines $x = -1$ and $x = 1$.

   The graph of the function f has vertical asymptotes $x = -1$ and $x = 1$. The reason for this is that at these values of x, the numerator of the function is not zero and the denominator is 0. So $x = -1$ and $x = 1$ are not in the domain of this function. In general, if a function is a quotient of two functions, then there will be a vertical asymptote for those values of x for which the numerator is not zero and the denominator is zero. We will see this for the tangent, cotangent, secant, and cosecant functions.

2. How is the tangent function defined? Complete the following: For each real number x with $\cos(t) \neq 0$, $\tan(t) =$ ______________.

3. Use a graphing utility to draw the graph of $y = \tan(t)$ using $-\pi \le t \le \pi$ and $-10 \le y \le 10$.

4. What are some of the vertical asymptotes of the graph of the function $y = \tan(t)$? What appears to be the range of the tangent function?

The Graph of the Tangent Function

The graph of the tangent function is very different than the graphs of the sine and cosine functions. One reason is that because $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$, there are values of t for which $\tan(t)$ is not defined. We have seen that

The domain of the tangent function is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

In particular, the real numbers $\dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$ are not in the domain of the tangent function. So the graph of the tangent function will have vertical asymptotes at $t = \dfrac{\pi}{2}$ and $t = -\dfrac{\pi}{2}$ (as well as at other values). We should have observed this in the beginning activity.

So to draw an accurate graph of the tangent function, it will be necessary to understand the behavior of the tangent near the points that are not in its domain. We now investigate the behavior of the tangent for points whose values of t that are slightly less than $\frac{\pi}{2}$ and for points whose values of t that are slightly greater than $-\frac{\pi}{2}$. Using a calculator, we can obtain the values shown in Table 2.4.

t	$\tan(t)$	t	$\tan(t)$
$\frac{\pi}{2} - 0.1$	9.966644423	$-\frac{\pi}{2} + 0.1$	-9.966644423
$\frac{\pi}{2} - 0.01$	99.99666664	$-\frac{\pi}{2} + 0.01$	-99.99666664
$\frac{\pi}{2} - 0.001$	999.9996667	$-\frac{\pi}{2} + 0.001$	-999.9996667
$\frac{\pi}{2} - 0.0001$	9999.999967	$-\frac{\pi}{2} + 0.0001$	-9999.999967

Table 2.4: Table of Values for the Tangent Function

So as the input t gets close to $\frac{\pi}{2}$ but stays less than $\frac{\pi}{2}$, the values of $\tan(t)$ are getting larger and larger, seemingly without bound. Similarly, as the input t gets close to $-\frac{\pi}{2}$ but stays greater than $-\frac{\pi}{2}$, the values of $\tan(t)$ are getting farther and farther away from 0 in the negative direction, seemingly without bound. We can see this in the definition of the tangent: as t gets close to $\frac{\pi}{2}$ from the left, $\cos(t)$ gets close to 0 and $\sin(t)$ gets close to 1. Now $\tan(t) = \frac{\sin(t)}{\cos(t)}$ and fractions where the numerator is close to 1 and the denominator close to 0 have very large values. Similarly, as t gets close to $-\frac{\pi}{2}$ from the right, $\cos(t)$ gets close to 0 (but is negative) and $\sin(t)$ gets close to 1. Fractions where the numerator is close to 1 and the denominator close to 0, but negative, are very large (in magnitude) negative numbers.

Progress Check 2.21 (The Graph of the Tangent Function)

1. Use a graphing utility to draw the graph of $y = \tan(t)$ using $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $-10 \le y \le 10$.

2. Use a graphing utility to draw the graph of $y = \tan(t)$ using $-\frac{3\pi}{2} \le t \le \frac{3\pi}{2}$ and $-10 \le y \le 10$.

3. Are these graphs consistent with the information we have discussed about vertical asymptotes for the tangent function?

4. What appears to be the range of the tangent function?

5. What appears to be the period of the tangent function?

Activity 2.22 (The Tangent Function and the Unit Circle)
The diagram in Figure 2.26 can be used to show how $\tan(t)$ is related to the unit circle definitions of $\cos(t)$ and $\sin(t)$.

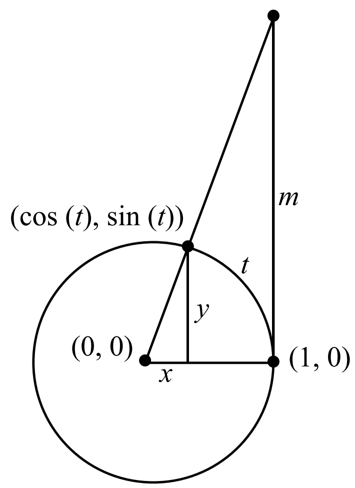


Figure 2.26: Illustrating $\tan(t)$ with the Unit Circle

In the diagram, an arc of length t is drawn and $\tan(t) = \dfrac{\cos(t)}{\sin(t)} = \dfrac{y}{x}$. This gives the slope of the line that goes through the points $(0, 0)$ and $(\cos(t), \sin(t))$. The vertical line through the point $(1, 0)$ intersects this line at the point $(1, m)$. This means that the slope of this line is also $\dfrac{m}{1}$ and hence, we see that

$$\tan t = \frac{\cos t}{\sin t} = m.$$

Now use the Geogebra applet *Tangent Graph Generator* to see how this information can be used to help see how the graph of the tangent function can be generated using the ideas in Figure 2.26. The web address is http://gvsu.edu/s/Zm

Effects of Constants on the Graphs of the Tangent Function

There are similarities and some differences in the methods of drawing the graph of a function of the form $y = A\tan(B(t - C)) + D$ and drawing the graph of a function of the form $y = A\sin(B(t - C)) + D$. See page 101 for a summary of the effects of the parameters A, B, C, and D on the graph of a sinusoidal function.

One of the differences in dealing with a tangent (secant, cotangent, or cosecant) function is that we do not use the terminology that is specific to sinusoidal waves. In particular, we will not use the terms amplitude and phase shift. Instead of amplitude, we use the more general term vertical stretch (or vertical compression), and instead of phase shift, we use the more general term horizontal shift. We will explore this is the following progress check.

Progress Check 2.23 Effects of Parameters on a Tangent Function

Consider the function whose equation is $y = 3\tan\left(2\left(x - \frac{\pi}{8}\right)\right) + 1$. Even if we use a graphing utility to draw the graph, we should answer the following questions first in order to get a reasonable viewing window for the graphing utility. It might be a good idea to use a method similar to what we would use if we were graphing $y = 3\sin\left(2\left(x - \frac{\pi}{8}\right)\right) + 1$.

1. We know that for the sinusoid, the period is $\dfrac{2\pi}{2}$. However, the period of the tangent function is π. So what will be the period of $y = 3\tan\left(2\left(x - \frac{\pi}{8}\right)\right) + 1$?
2. For the sinusoid, the amplitude is 3. However, we do not use the term "amplitude" for the tangent. So what is the effect of the parameter 3 on the graph of $y = 3\tan\left(2\left(x - \frac{\pi}{8}\right)\right) + 1$?
3. For the sinusoid, the phase shift is $\dfrac{\pi}{8}$. However, we do not use the term "phase shift" for the tangent. So what is the effect of the parameter $\dfrac{\pi}{8}$ on the graph of $y = 3\tan\left(2\left(x - \frac{\pi}{8}\right)\right) + 1$?
4. Use a graphing utility to draw the graph of this function for one complete period. Use the period of the function that contains the number 0.

The Graph of the Secant Function

To understand the graph of the secant function, we need to recall the definition of the secant and the restrictions on its domain. If necessary, refer to Section 1.6 to complete the following progress check.

Progress Check 2.24 (The Secant Function)

1. How is the secant function defined?
2. What is the domain of the secant function?
3. Where will the graph of the secant function have vertical asymptotes?
4. What is the period of the secant function?

Activity 2.25 (The Graph of the Secant Function)

1. We will use the Geogebra Applet with the following web address:

   `http://gvsu.edu/s/Zn`

   This applet will show how the graph of the secant function is related to the graph of the cosine function. In the applet, the graph of $y = \cos(t)$ is shown and is left fixed. We generate points on the graph of $y = \sec(t)$ by using the slider for t. For each value of t, a vertical line is drawn from the point $(t, \cos(t))$ to the point $(t, \sec(t))$. Notice how these points indicate that the graph of the secant function has vertical asymptotes at $t = \dfrac{\pi}{2}$, $t = \dfrac{3\pi}{2}$, and $t = \dfrac{5\pi}{2}$.

2. Use a graphing utility to draw the graph of $y = \sec(x)$ using $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ and $-10 \leq y \leq 10$. **Note**: It may be necessary to use $\sec(x) = \dfrac{1}{\cos(x)}$.

3. Use a graphing utility to draw the graph of $y = \sec(x)$ using $-\dfrac{3\pi}{2} \leq x \leq \dfrac{3\pi}{2}$ and $-10 \leq y \leq 10$.

The work in Activity 2.25 and Figure 2.27 can be used to help answer the questions in Progress Check 2.26.

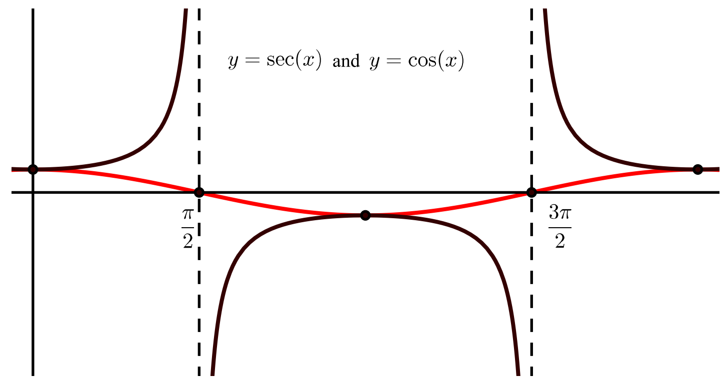


Figure 2.27: Graph of One Period of $y = \sec(x)$ with $0 \le x \le 2\pi$

Progress Check 2.26 (The Graph of the Secant Function)

1. Is the graph in Figure 2.27 consistent with the graphs from Activity 2.25?
2. Why is the graph of $y = \sec(x)$ above the x-axis when $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$?
3. Why is the graph of $y = \sec(x)$ below the x-axis when $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$?
4. What is the range of the secant function?

Summary of Section 2.4

In this section, we studied the following important concepts and ideas:

- **The Tangent Function**. Table 2.5 shows some of the important characteristics of the tangent function. We have already discussed most of these items, but the last two items in this table will be explored in Exercise (1) and Exercise (2).

  A graph of three periods of the tangent function is shown in Figure 2.28.

	$y = \tan(t)$	$y = \sec(t)$
period	π	2π
domain	real numbers t with $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k	
y-intercept	$(0, 0)$	$(0, 1)$
x-intercepts	$t = k\pi$, where k is some integer	none
symmetry	with respect to the origin	with respect to the y-axis

Table 2.5: Properties of the Tangent and Secant Functions

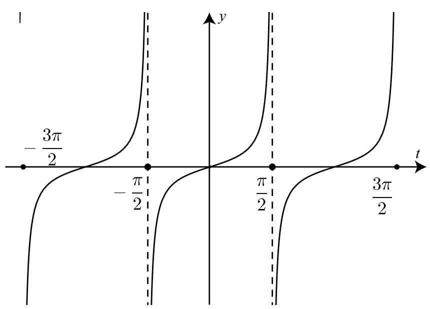


Figure 2.28: Graph of $y = \tan(t)$

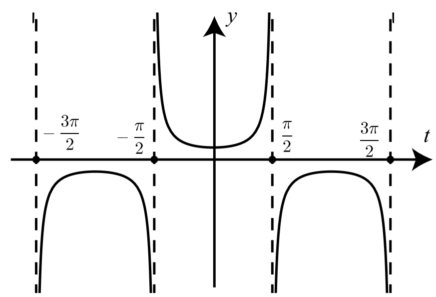


Figure 2.29: Graph of $y = \sec(t)$

- **The Secant Function**. Table 2.5 shows some of the important characteristics of the secant function. The symmetry of the secant function is explored in Exercise (3). Figure 2.29 shows a graph of the secant function.

- **The Cosecant Function**. The graph of the cosecant function is studied in a way that is similar to how we studied the graph of the secant function. This is done in Exercises (4), (5), and (6). Table 2.6 shows some of the important characteristics of the cosecant function. The symmetry of the cosecant function is explored in Exercise (3). Figure 2.30 shows a graph of the cosecant function.

- **The Cotangent Function**. The graph of the cosecant function is studied in a way that is similar to how we studied the graph of the tangent function. This is done in Exercises (7), (8), and (9). Table 2.6 shows some of the important characteristics of the cotangent function. The symmetry of the cotangent function is explored in Exercise (3). Figure 2.31 shows a graph of the cotangent function.

	$y = \csc(t)$	$y = \cot(t)$
period	2π	π
domain	real numbers t with $t \neq k\pi$ for all integers k	
range	$\|y\| \geq 1$	all real numbers
y-intercept	none	none
x-intercepts	none	$t = \dfrac{\pi}{2} + k\pi$, where k is an integer
symmetry	with respect to the origin	

Table 2.6: Properties of the Cosecant and Cotangent Functions

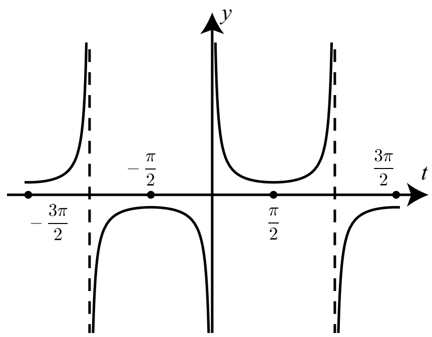


Figure 2.30: Graph of $y = \csc(t)$

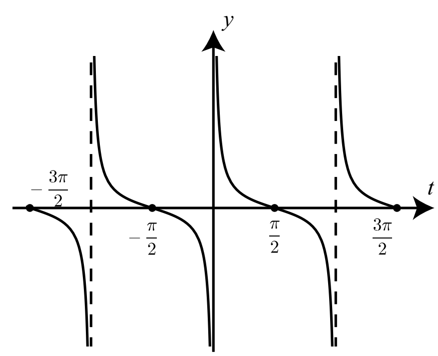


Figure 2.31: Graph of $y = \cot(t)$

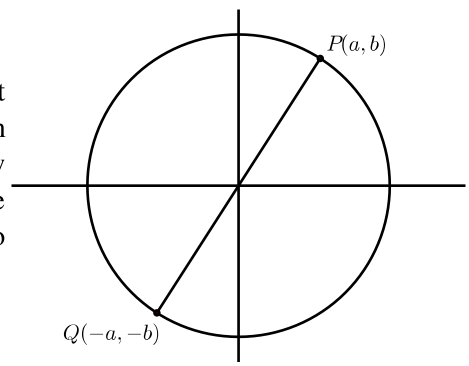
Exercises for Section 2.4

1. In this exercise, we will explore the period of the tangent function.

⋆ **(a)** Use the definition of the tangent function and the fact that the period of both the sine and cosine functions is equal to 2π to prove that for any real number t in the domain of the tangent function,

$$\tan(t + 2\pi) = \tan(t).$$

However, this does not prove that the period of the tangent function is equal to 2π. We will now show that the period is equal to π. The key to the proof is the diagram to the right.



Suppose that P is the terminal point of the arc t. So $\cos(t) = a$ and $\sin(t) = b$. The diagram shows a point Q that is the terminal point of the arc $t + \pi$. By the symmetry of the circle, we know that the point Q has coordinates $(-a, -b)$.

(b) Explain why $\cos(t + \pi) = -a$ and $\sin(t + \pi) = -b$.

(c) Use the information in part (a) and the definition of the tangent function to prove that $\tan(t + \pi) = \tan(t)$.

The diagram also indicates that the smallest positive value of p for which $\tan(t + p) = \tan(t)$ must be $p = \pi$. Hence, the period the tangent function is equal to π.

2. We have seen that $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ for every real number t. Now assume that t is a real number for which $\tan(t)$ is defined.

(a) Use the definition of the tangent function to write a formula for $\tan(-t)$ in terms of $\sin(-t)$ and $\cos(-t)$.

(b) Now use the negative arc identities for the cosine and sine functions to help prove that $\tan(-t) = -\tan(t)$. This is called the *negative arc identity for the tangent function*.

(**c**) Use the negative arc identity for the tangent function to explain why the graph of $y = \tan(t)$ is symmetric about the origin.

3. Use the negative arc identities for sine, cosine, and tangent to help prove the following negative arc identities for cosecant, secant, and cotangent.

⋆ **(a)** For every real number t for which $t \neq k\pi$ for every integer k, $\csc(-t) = -\csc(t)$.

(b) For every real number t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k, $\sec(-t) = \sec(t)$.

(c) For every real number t for which $t \neq k\pi$ for every integer k, $\cot(-t) = -\cot(t)$.

4. The Cosecant Function. If necessary, refer to Section 1.6 to answer the following questions.

(**a**) How is the cosecant function defined?

(**b**) What is the domain of the cosecant function?

(**c**) Where will the graph of the cosecant function have vertical asymptotes?

(**d**) What is the period of the cosecant function?

5. Exploring the Graph of the Cosecant Function.

(**a**) Use the Geogebra Applet with the following web address to explore the relationship between the graph of the cosecant function and the sine function.

`http://gvsu.edu/s/0bH`

In the applet, the graph of $y = \sin(t)$ is shown and is left fixed. Points on the graph of $y = \csc(t)$ are generated by using the slider for t. For each value of t, a vertical line is drawn from the point $(t, \sin(t))$ to the point $(t, \csc(t))$. Notice how these points indicate that the graph of the cosecant function has vertical asymptotes at $t = 0$, $t = \pi$, and $t = 2\pi$.

(**b**) Use a graphing utility to draw the graph of $y = \csc(x)$ using $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ and $-10 \leq y \leq 10$. **Note**: It may be necessary to use $\csc(x) = \dfrac{1}{\sin(x)}$.

(**c**) Use a graphing utility to draw the graph of $y = \csc(x)$ using $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ and $-10 \leq y \leq 10$.

6. **The Graph of the Cosecant Function**.

    (**a**) Why does the graph of $y = \csc(x)$ have vertical asymptotes at $x = 0$, $x = \pi$, and $x = 2\pi$? What is the domain of the cosecant function?

    (**b**) Why is the graph of $y = \csc(x)$ above the x-axis when $0 < x < \pi$?

    (**c**) Why is the graph of $y = \csc(x)$ below the x-axis when $\pi < x < 2\pi$?

    (**d**) What is the range of the cosecant function?

7. **The Cotangent Function**. If necessary, refer to Section 1.6 to answer the following questions.

    (**a**) How is the cotangent function defined?

    (**b**) What is the domain of the cotangent function?

    (**c**) Where will the graph of the cotangent function have vertical asymptotes?

    (**d**) What is the period of the cotangent function?

8. **Exploring the Graph of the Cotangent Function**.

    (**a**) Use a graphing utility to draw the graph of $y = \cot(x)$ using $-\pi \leq x \leq \pi$ and $-10 \leq y \leq 10$. **Note**: It may be necessary to use $\cot(x) = \frac{1}{\tan(x)}$.

    (**b**) Use a graphing utility to draw the graph of $y = \cot(x)$ using $-2\pi \leq x \leq 2\pi$ and $-10 \leq y \leq 10$.

9. **The Graph of the Cotangent Function**.

    (**a**) Why does the graph of $y = \cot(x)$ have vertical asymptotes at $x = 0$, $x = \pi$, and $x = 2\pi$? What is the domain of the cotangent function?

    (**b**) Why is the graph of $y = \cot(x)$ above the x-axis when $0 < x < \frac{\pi}{2}$ and when $\pi < x < \frac{3\pi}{2}$?

    (**c**) Why is the graph of $y = \cot(x)$ below the x-axis when $\frac{\pi}{2} < x < \pi$ and when $\frac{3\pi}{2} < x < 2\pi$?

    (**d**) What is the range of the cotangent function?

2.5 Inverse Trigonometric Functions

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- Why doesn't the sine function really have an inverse? What do we mean, then, by the inverse sine function? In other words, how is the inverse sine function defined?
- How is the inverse cosine function defined?
- How is the inverse tangent function defined?

Beginning Activity

1. If $y = 5x + 7$ and $y = 4$, what is the value of x?

2. If $y = \sqrt{x}$ and $y = 2.5$, what is the value of x?

3. If $y = x^2$ and $y = 25$, what are the possible values of x?

4. If $y = \sin(x)$ and $y = \dfrac{1}{2}$, find two values for x with $0 \leq x \leq 2\pi$.

Introduction

The work in the beginning activity illustrates the general problem that if we are given a function f and $y = f(x)$, can we find the values of x if we know the value of y. In effect, this means that if we know the value of y, can we solve for the value of x? For the first problem, we can substitute $y = 4$ into $y = 5x + 7$ and solve for x. This gives

$$\begin{aligned} 4 &= 5x + 7 \\ -3 &= 5x \\ x &= \frac{-3}{5} \end{aligned}$$

For the second and third problems, we have

$$\begin{aligned} 2.5 &= \sqrt{x} & 25 &= x^2 \\ 2.5^2 &= \left(\sqrt{x}\right)^2 & x &= \pm\sqrt{25} \\ x &= 6.25 & x &= \pm 5 \end{aligned}$$

The work with the equation $x^2 = 25$ shows that we can have more than one solution for this type of problem. With trigonometric functions, we can even have more solutions. For example, if $y = \sin(x)$ and $y = \frac{1}{2}$, we have

$$\sin(x) = \frac{1}{2}.$$

If we restrict the values of x to $0 \leq x \leq 2\pi$, there will be two solutions as shown in Figure 2.32

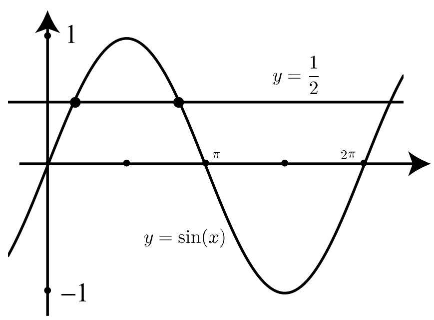


Figure 2.32: Graph Showing $\sin(x) = \frac{1}{2}$

From our knowledge of the common arcs and reference arcs, these two solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. In addition, the periodic nature of the sine function tells us that if there are no restrictions on x, there will be infinitely many solutions of the equation $\sin(x) = \frac{1}{2}$. What we want to develop is a method to indicate exactly one of these solutions. But which one do we choose?

We have done something like this when we solve an equation such as $x^2 = 25$. There are two solutions to this equation, but we have a function (the square root function) that gives us exactly one of these two functions. So when we write $x = \sqrt{25} = 5$, we are specifying only the positive solution of the equation. If we want the other solution, we have to write $x = -\sqrt{25} = -5$. Notice that we used the square root function to designate the "simpler" of the two functions, namely the positive solution.

For the sine function, what we want is an inverse sine function that does just what the name suggests – uniquely reverses what the sine function does. That is, the inverse sine function takes a value from the range of the sine function and gives us exactly one arc whose sine has that value. We will try to do this in as simple of a manner as possible. (It may sometimes be hard to believe, but mathematicians generally do try to keep things simple.) To be more specific, if we have $y = \sin(x)$, we want to be able to specify any value for y with $-1 \le y \le 1$ and obtain one value for x. We will choose the value for x that is as close to 0 as possible. (Keep it simple.)

So to ensure that there is only one solution, we will restrict the graph of $y = \sin(x)$ to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. This also guarantees that $-1 \le y \le 1$ as shown in Figure 2.33.

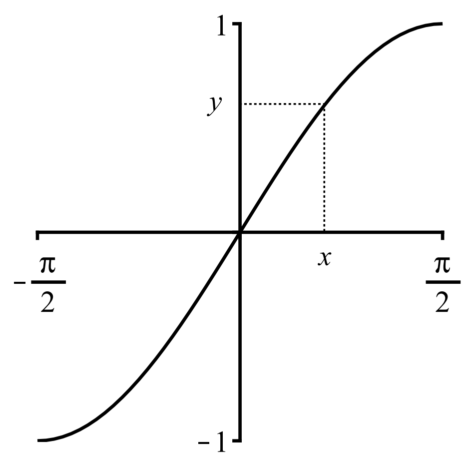


Figure 2.33: Graph of $y = \sin(x)$ restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

As is illustrated in Figure 2.33, for each value of y with $-1 \le y \le 1$, there is exactly one value of x with $\sin(x) = y$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Definition. The **inverse sine function** (denoted by arcsin or $\sin^{-1}$), is defined as follows:
For $-1 \leq y \leq 1$,

$$t = \arcsin(y) \quad \text{or} \quad t = \sin^{-1}(y)$$

means that

$$y = \sin(t) \quad \text{and} \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Caution. Either notation may be used for the arcsine function. That is, $\arcsin(y)$ and $\sin^{-1}(y)$ mean the same thing. However, the notation $\sin^{-1}$ does not mean the reciprocal of the sine but rather the inverse of the sine with a restricted domain. It is very important to remember the facts that the domain of the inverse sine is the interval $[-1, 1]$ and the range of the inverse sine is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note: Some people prefer using $t = \arcsin(y)$ instead of $t = \sin^{-1}(y)$ since it can be a reminder of what the notation means. The equation $t = \arcsin(y)$ is an abbreviation for

$$t \text{ is the } \textbf{arc} \text{ with } \textbf{sine} \text{ value } y \text{ and } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

It is important to keep writing the restriction $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ since it is important to realize that $\arcsin(y)$ function gives only one arc whose sine value is y and t must be in this interval.

Example 2.27 (Inverse Sine Function)

We will determine the exact value of $\arcsin\left(\frac{\sqrt{3}}{2}\right)$. So we let $t = \arcsin\left(\frac{\sqrt{3}}{2}\right)$. This means that

$$\sin(t) = \frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

That is, we are trying to find the arc t whose sine is $\frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Using our knowledge of sine values for common arcs, we notice that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and so we conclude that $t = \dfrac{\pi}{3}$ or that

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

This is illustrated graphically in Figure 2.34.

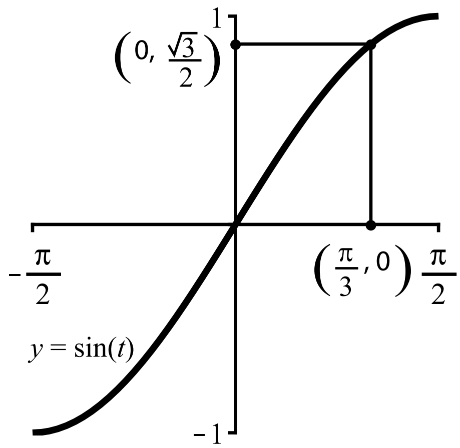


Figure 2.34: Graphical Version of $\arcsin\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

Note: Most calculators and graphing utilities can calculate approximate values for the inverse sine function. On calculators, it is often the $\sin^{-1}$ key and for many computer programs, it is necessary to type "arcsin." Using a calculator, we see that $\arcsin\left(\dfrac{\sqrt{3}}{2}\right) \approx 1.04720$, which is a decimal approximation for $\dfrac{\pi}{3}$.

Progress Check 2.28 (Calculating Values for the Inverse Sine Function)
Determine the exact value of each of the following. You may check your results with a calculator.

1. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

2. $\sin^{-1}\left(\dfrac{1}{2}\right)$

3. $\arcsin(-1)$

4. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$

In the next progress check, we will use the inverse sine function in two-step calculations. Please pay attention to the results that are obtained.

Progress Check 2.29 (Calculations Involving the Inverse Sine Function)
Determine the exact value of each of the following. You may check your results with a calculator.

1. $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$
2. $\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right)$
3. $\sin\left(\sin^{-1}\left(\frac{2}{5}\right)\right)$
4. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

The work in Progress Check 2.29 illustrates some important properties of the inverse since function when it is composed with the sine function. This property is that in some sense, the inverse sine and the sine functions "undo" each other. To see what this means, we let $y = \sin(t)$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Then

$$\sin^{-1}(\sin(t)) = \sin^{-1}(y) = t$$

by definition. This means that if we apply the sine, then the inverse sine to an arc between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, we get back the arc. This is what we mean when we say the inverse sine undoes the sine.

Similarly, if $t = \sin^{-1}(y)$ for some y with $-1 \leq y \leq 1$, then

$$\sin(\sin^{-1}(y)) = \sin(t) = y$$

by definition. So the sine also undoes the inverse sine as well. We summarize these two results as follows:

Properties of the Inverse Sine Function

- For each t in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\sin^{-1}(\sin(t)) = t \quad \text{or} \quad \arcsin(\sin(t)) = t.$$

- For each y in the closed interval $[-1, 1]$,

$$\sin\left(\sin^{-1}(y)\right) = y \quad \text{or} \quad \sin(\arcsin(y)) = y.$$

The Inverse Cosine and Inverse Tangent Functions

In a manner similar to how we defined the inverse sine function, we can define the inverse cosine and the inverse tangent functions. The key is to restrict the domain of the corresponding circular function so that we obtain the graph of a one-to-one function. So we will use $y = \cos(t)$ with $0 \le t \le \pi$ and $y = \tan(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$ as is illustrated in Figure 2.35.

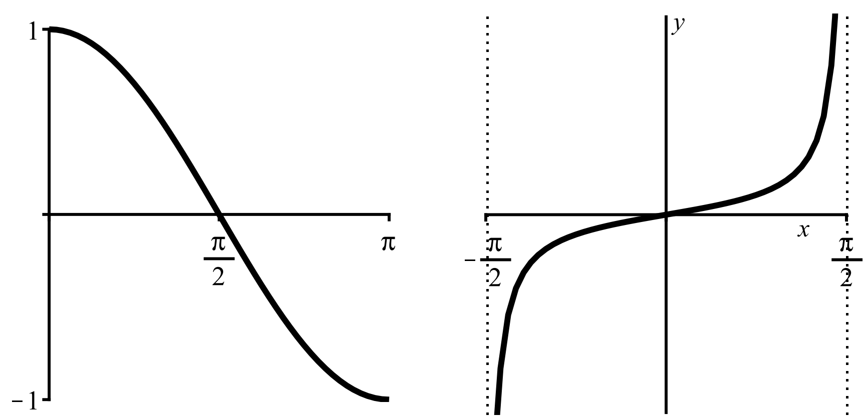


Figure 2.35: Graph of $y = \cos(t)$ for $0 \le t \le \pi$ and Graph of $y = \tan(t)$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Note: We do not use the interval $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$ for the cosine function since the cosine function is not one-to-one on that interval. In addition, the interval for the tangent function does not contain the endpoints since the tangent function is not defined at $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

Using these domains, we now define the inverse functions for cosine and tangent.

Definition. We define the **inverse cosine function**, arccos or $\cos^{-1}$, as follows:
For $-1 \leq y \leq 1$,

$$t = \arccos(y) \quad \text{or} \quad t = \cos^{-1}(y)$$

means that

$$y = \cos(t) \quad \text{and} \quad 0 \leq t \leq \pi.$$

We define the **inverse tangent function**, arctan or $\tan^{-1}$, as follows:
For $t \in \mathbb{R}$,

$$t = \arctan(y) \quad \text{or} \quad y = \tan^{-1}(y)$$

means that

$$y = \tan(t) \quad \text{and} \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Example 2.30 (An Example of Inverse Cosine)
The equation $y = \arccos\left(-\frac{1}{2}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$ means that

$$\cos(y) = -\frac{1}{2} \quad \text{and} \quad 0 \leq y \leq \pi.$$

That is, we are trying to find the arc y whose cosine is $-\frac{1}{2}$ and $0 \leq y \leq \pi$. Using our knowledge of cosine values for common arcs, we notice that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So we conclude that the reference angle $\hat{y}$ for y is $\hat{y} = \frac{\pi}{3}$. Since y must be in the second quadrant, we conclude that $y = \pi - \frac{\pi}{3}$ or $y = \frac{2\pi}{3}$. So

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

This can be checked using a calculator and is illustrated in Figure 2.36.

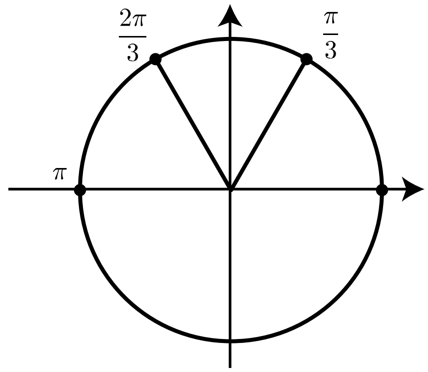


Figure 2.36: Diagram Used for the Inverse Cosine of $-\dfrac{1}{2}$.

Progress Check 2.31 (Inverse Cosine and Inverse Tangent Functions)
Determine the exact value of each of the following. You may check your results with a calculator.

1. $\cos\left(\cos^{-1}\left(\dfrac{1}{2}\right)\right)$
2. $\arccos\left(\cos\left(\dfrac{\pi}{4}\right)\right)$
3. $\arccos\left(\cos\left(\dfrac{-\pi}{4}\right)\right)$
4. $\tan^{-1}\left(\tan\left(\dfrac{5\pi}{4}\right)\right)$

The work in Progress Check 2.31 illustrates some important properties of the inverse cosine and inverse tangent functions similar to the properties of the inverse sine function on page 147.

Properties of the Inverse Cosine Function

- For each t in the closed interval $[0, \pi]$,

$$\cos^{-1}(\cos(t)) = t \quad \text{or} \quad \arccos(\cos(t)) = t.$$

- For each y in the closed interval $[-1, 1]$,

$$\cos\left(\cos^{-1}(y)\right) = y \quad \text{or} \quad \cos(\arccos(y)) = y.$$

Properties of the Inverse Tangent Function

- For each t in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}(\tan(t)) = t \quad \text{or} \quad \arctan(\tan(t)) = t.$$

- For each real number y,

$$\tan\left(\tan^{-1}(y)\right) = y \quad \text{or} \quad \tan(\arctan(y)) = y.$$

The justification for these properties is included in the exercises.

Progress Check 2.32 (Inverse Trigonometric Functions)
Determine the exact value of each of the following and check them using a calculator.

1. $y = \arccos(1)$
2. $y = \tan^{-1}\left(\sqrt{3}\right)$
3. $y = \arctan(-1)$
4. $y = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
5. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$
6. $\tan\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right)$
7. $\arccos\left(\sin\left(\frac{\pi}{6}\right)\right)$

When we evaluate an expression such as $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$ in Progress Check 2.32, we can use the fact that it is possible to determine the exact value of $\arccos\left(-\frac{1}{2}\right)$ to complete the problem. If we are given a similar problem but do not know the exact value of an inverse trigonometric function, we can often use the Pythagorean Identity to help. We will do this in the next progress check.

Progress Check 2.33 (Using the Pythagorean Identity)

1. Determine the exact value of $\sin\left(\arccos\left(\frac{1}{3}\right)\right)$. Following is a suggested way to start this. Since we do not know the exact value of $\arccos\left(\frac{1}{3}\right)$, we

start by letting $t = \arccos\left(\dfrac{1}{3}\right)$. We then know that

$$\cos(t) = \frac{1}{3} \quad \text{and} \quad 0 \leq t \leq \pi.$$

Notice that $\sin(t) = \sin\left(\arccos\left(\dfrac{1}{3}\right)\right)$. So to complete the problem, determine the exact value of $\sin(t)$ using the Pythagorean Identity keeping in mind that $0 \leq t \leq \pi$.

2. Determine the exact value of $\cos\left(\arcsin\left(-\dfrac{4}{7}\right)\right)$.

Summary of Section 2.5

In this section, we studied the following important concepts and ideas:

- **The Inverse Sine Function** uniquely reverses what the sine function does. The inverse sine function takes a value y from the range of the sine function and gives us exactly one real number t whose sine is equal to y. That is, if y is a real number and $-1 \leq y \leq 1$, then

$$\sin^{-1}(y) = t \text{ means that } \sin(t) = y \text{ and } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

  In addition, the inverse sine function satisfies the following important properties:

  * For each t in the closed interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,

$$\sin^{-1}(\sin(t)) = t.$$

  * For each y in the closed interval $[-1, 1]$,

$$\sin\left(\sin^{-1}(y)\right) = y.$$

- **The Inverse cosine Function** uniquely reverses what the cosine function does. The inverse cosine function takes a value y from the range of the cosine function and gives us exactly one real number t whose cosine is equal to y. That is, if y is a real number and $-1 \leq y \leq 1$, then

$$\cos^{-1}(y) = t \text{ means that } \cos(t) = y \text{ and } 0 \leq t \leq \pi.$$

In addition, the inverse cosine function satisfies the following important properties:

* For each t in the closed interval $[0, \pi]$,

$$\cos^{-1}(\cos(t)) = t.$$

* For each y in the closed interval $[-1, 1]$,

$$\cos\left(\cos^{-1}(y)\right) = y.$$

- **The Inverse Tangent Function** uniquely reverses what the tangent function does. The inverse tangent function takes a value y from the range of the tangent function and gives us exactly one real number t whose tangent is equal to y. That is, if y is a real number, then

$$\tan^{-1}(y) = t \text{ means that } \tan(t) = y \text{ and } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

In addition, the inverse tangent function satisfies the following important properties:

* For each t in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}(\tan(t)) = t.$$

* For each real number y,

$$\tan\left(\tan^{-1}(y)\right) = y.$$

Exercises for Section 2.5

1. Rewrite each of the following using the corresponding trigonometric function for the inverse trigonometric function. Then determine the exact value of the inverse trigonometric function.

⋆ **(a)** $t = \arcsin\left(\dfrac{\sqrt{2}}{2}\right)$ ⋆ **(b)** $t = \arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$

(c) $t = \arccos\left(\dfrac{\sqrt{2}}{2}\right)$

⋆ (d) $t = \arccos\left(-\dfrac{\sqrt{2}}{2}\right)$

(e) $y = \tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right)$

⋆ (f) $y = \tan^{-1}\left(\dfrac{-\sqrt{3}}{3}\right)$

(g) $y = \cos^{-1}(0)$

⋆ (h) $t = \arctan(0)$

(i) $y = \sin^{-1}\left(-\dfrac{1}{2}\right)$

⋆ (j) $y = \cos^{-1}\left(-\dfrac{1}{2}\right)$

2. Determine the exact value of each of the following expressions.

⋆ (a) $\sin\left(\sin^{-1}(1)\right)$

⋆ (b) $\sin^{-1}\left(\sin\left(\dfrac{\pi}{3}\right)\right)$

(c) $\cos^{-1}\left(\sin\left(\dfrac{\pi}{3}\right)\right)$

(d) $\sin^{-1}\left(\sin\left(-\dfrac{\pi}{3}\right)\right)$

⋆ (e) $\cos^{-1}\left(\cos\left(-\dfrac{\pi}{3}\right)\right)$

⋆ (f) $\arcsin\left(\sin\left(\dfrac{2\pi}{3}\right)\right)$

(g) $\tan\left(\arctan(1)\right)$

(h) $\arctan\left(\tan\left(\dfrac{\pi}{4}\right)\right)$

⋆ (i) $\arctan\left(\tan\left(\dfrac{3\pi}{4}\right)\right)$

3. Determine the exact value of each of the following expressions.

⋆ (a) $\cos\left(\arcsin\left(\dfrac{2}{5}\right)\right)$

⋆ (b) $\sin\left(\arccos\left(-\dfrac{2}{3}\right)\right)$

⋆ (c) $\tan\left(\arcsin\left(\dfrac{1}{3}\right)\right)$

(d) $\cos\left(\arcsin\left(-\dfrac{2}{5}\right)\right)$

(e) $\tan\left(\arccos\left(-\dfrac{2}{9}\right)\right)$

4. This exercise provides a justifiction for the properties of the inverse cosine function on page 150. Let t be a real number in the closed interval $[0, \pi]$ and let

$$y = \cos(t). \tag{1}$$

We then see that $-1 \leq y \leq 1$ and

$$\cos^{-1}(y) = t. \tag{2}$$

(a) Use equations (1) and (2) to rewrite the expression $\cos^{-1}(\cos(t))$.

(**b**) Use equations (1) and (2) to rewrite the expression $\cos\left(\cos^{-1}(y)\right)$.

5. This exercise provides a justifiction for the properties of the inverse tangent function on page 151. Let t be a real number in the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and let

$$y = \tan(t). \tag{3}$$

We then see that y is a real number and

$$\tan^{-1}(y) = t. \tag{4}$$

(**a**) Use equations (3) and (4) to rewrite the expression $\tan^{-1}(\tan(t))$.

(**b**) Use equations (3) and (4) to rewrite the expression $\tan\left(\tan^{-1}(y)\right)$.

2.6 Solving Trigonometric Equations

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

For these questions, we let q be a real number with $-1 \leq q \leq 1$ and let r be a real number.

- How can an inverse trigonometric function be used to determine one solution of an equation of the form $\sin(x) = q$, $\cos(x) = q$, or $\tan(x) = r$?
- How can properties of the trigonometric functions be used to determine all solutions of an equation of the form $\sin(x) = q$, $\cos(x) = q$, or $\tan(x) = r$ within one complete period of the trigonometric function?
- How can we use the period of a trigonometric function to determine a formula for the solutions of an equation of the form $\sin(x) = q$, $\cos(x) = q$, or $\tan(x) = r$?

Recall that a mathematical *equation* like $x^2 = 1$ is a relation between two expressions that may be true for some values of the variable while an *identity* like $\cos(-x) = \cos(x)$ is an equation that is true for all allowable values of the variable. So an identity is a special type of equation. Equations that are not identities are also called *conditional equations* because they are not valid for all allowable values of the variable. To *solve* an equation means to find all of the values for the variables that make the two expressions on either side of the equation equal to each other. We solved algebraic equations in algebra and now we will solve trigonometric equations.

A **trigonometric equation** is an equation that involves trigonometric functions. We have already used graphical methods to approximate solutions of trigonometric equations. In Example 2.17 on page 112, we used the function

$$V(t) = 35\cos\left(\frac{5\pi}{3}t\right) + 105$$

as a model for the amount of blood in the heart. For this function, t is measured in seconds since the heart was full and $V(t)$ is measured in milliliters. To determine the times when there are 100 milliliters of blood in the heart, we needed to solve the equation

$$35\cos\left(\frac{5\pi}{3}t\right) + 105 = 100.$$

At that time, we used the "intersect" capability of a graphing utility to determine some solutions of this equation. In this section, we will learn how to use the inverse cosine function and properties of the cosine function to determine the solutions of this equation. We begin by first studying simpler equations.

Beginning Activity

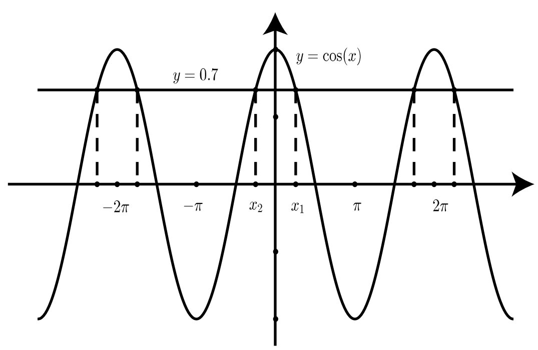
Use a graphing utility to draw the graphs of $y = \cos(x)$ and $y = 0.7$ on the same axes using $-\pi \le x \le \pi$ and $-1.2 \le y \le 1.2$. Use the graphing utility to find the points of intersection of these two graphs and to determine the solutions of the equation $\cos(x) = 0.7$ with $-\pi \le x \le \pi$.

In the beginning activity, we should have determined the following approximations for solutions of the equation $\cos(x) = 0.7$.

$$x_1 \approx 0.79540, x_2 \approx -0.79540.$$

These approximations have been rounded to five decimal places.

The graph to the right shows the two graphs using $-3\pi \le x \le 3\pi$. The solutions x_1 and x_2 are shown on the graph. As can be seen, the graph shows x_1 and x_2 and four other solutions to the equation $\cos(x) = 0.7$. In fact, if we imagine the graph extended indefinitely to the left and to the right, we can see that there are infinitely many solutions for this equation.



This is where we can use the fact that the period of the cosine function is 2π. The other solutions differ from x_1 or x_2 by an integer multiple of the period of 2π. We can represent an integer multiple of 2π by $k(2\pi)$ for some integer k. So we say that any solution of the equation $\cos(x) = 0.7$ can be approximated by

$$x \approx 0.79540 + k(2\pi) \quad \text{or} \quad x \approx -0.79540 + k(2\pi).$$

For example, if we use $k = 4$, we see that

$$x \approx 25.92814 \quad \text{or} \quad x \approx 24.33734.$$

We can use a calculator to check that for both values, $\cos(x) = 0.7$.

A Strategy for Solving a Trigonometric Equation

The example using the equation $\cos(x) = 0.7$ was designed to illustrate the fact that if there are no restrictions placed on the unknown x, then there can be infinitely many solutions for an equation of the form

"some trigonometric function of x" = a number.

A general strategy to solve such equations is:

- Find all solutions of the equation within one period of the function. This is often done by using properties of the trigonometric function. Quite often, there will be two solutions within a single period.
- Use the period of the function to express formulas for all solutions by adding integer multiples of the period to each solution found in the first step. For example, if the function has a period of 2π and x_1 and x_2 are the only two solutions in a complete period, then we would write the solutions for the equation as

  $$x = x_1 + k(2\pi), \quad x = x_2 + k(2\pi), \text{ where } k \text{ is an integer.}$$

  Note: Instead of writing "k is an integer," we could write $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Progress Check 2.34 (Solving a Trigonometric Equation)
Use a graph to approximate the solutions (rounded to four decimal places) of the equation $\sin(x) = -0.6$ on the interval $-\pi \leq x \leq \pi$. Then use the period of the sine function to write formulas that can be used to approximate any solution of this equation.

Using Inverse Functions to Solve Trigonometric Equations

Although we can use a graphing utility to determine approximations for solutions to many equations, we often need to have some notation to indicate specific numbers (that are often solutions of equations). We have already seen this in previous mathematics courses. For example, we use the notation $\sqrt{20}$ to represent the positive real number whose square is equal to 20. We can use this to say that the two solutions of the equation $x^2 = 20$ are

$$x = \sqrt{20} \quad \text{and} \quad x = -\sqrt{20}.$$

Notice that there are two solutions of the equation but $\sqrt{20}$ represents only one of those solutions. We will now learn how to use the inverse trigonometric functions to do something similar for trigonometric equations. One big difference is that most trigonometric equations will have infinitely many solutions instead of just two. We will use the inverse trigonometric functions to represent one solution of an equation and then learn how to represent all of the solutions in terms of this one solution. We will first show how this is done with the equation $\cos(x) = 0.7$ from the beginning activity for this section.

Example 2.35 (Solving an Equation Involving the Cosine Function)
For the equation $\cos(x) = 0.7$, we first use the result about the inverse cosine function on page 150, which states that for t in the closed interval $[0, \pi]$,

$$\cos^{-1}(\cos(t)) = t.$$

So we "apply the inverse cosine function" to both sides of the equation $\cos(x) = 0.7$. This gives:

$$\begin{aligned} \cos(x) &= 0.7 \\ \cos^{-1}(\cos(x) &= \cos^{-1}(0.7) \\ x &= \cos^{-1}(0.7) \end{aligned}$$

Another thing we must remember is that this gives the one solution for the equation that is in interval $[0, \pi]$. Before we use the periodic property, we need to determine the other solutions for the equation in one complete period of the cosine function. We can use the interval $[0, 2\pi]$ but it is easier to use the interval $[-\pi, \pi]$. One reason for this is the following so-called "negative arc identity" stated on page 82.

$$\cos(-x) = \cos(x) \text{ for every real number } x.$$

Hence, since one solution for the equation is $x = \cos^{-1}(0.7)$, another solution is $x = -\cos^{-1}(0.7)$. This means that the two solutions of the equation $x = \cos(x)$ on the interval $[-\pi, \pi]$ are

$$x = \cos^{-1}(0.7) \quad \text{and} \quad x = -\cos^{-1}(0.7).$$

It can be verified that the equation $\cos(x) = 0.7$ has two solutions on the interval $[-\pi, \pi]$ by drawing the graphs of $y = \cos(x)$ and $y = 0.7$ on the interval $[-\pi, \pi]$. So if we restrict ourselves to this interval, we have something very much like solving the equation $x^2 = 20$ in that there are two solutions that are negatives of each other. The main difference now is that the trigonometric equation has infinitely many solutions and as before, we now use the periodic property of the cosine function. Since the period is 2π, just like with the numerical approximations from the beginning activity, we can say that any solution of the equation $\cos(x) = 0.7$ will be of the form

$$x = \cos^{-1}(0.7) + k(2\pi) \quad \text{or} \quad x = -\cos^{-1}(0.7) + k(2\pi),$$

where k is some integer.

Progress Check 2.36 (Solving an Equation)
Determine all solutions of the equation $4\cos(x) + 3 = 2$ in the interval $[-\pi, \pi]$. Then use the periodic property of the cosine function to write formulas that can be used to generate all the solutions of this equation. **Hint**: First use algebra to rewrite the equation in the form $\cos(x) =$ "some number".

The previous examples have shown that when using the inverse cosine function to solve equations of the form $\cos(x) =$ a number, it is easier to use the interval $[-\pi, \pi]$ rather than the interval $[0, 2\pi]$. This is not necessarily true when using the inverse sine function since the inverse sine function gives a value in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. However, to keep things similar, we will continue to use the interval $[-\pi, \pi]$ as the complete period for the sine (or cosine) function. For the inverse sine, we use the following property stated on page 147.

For each t in the closed interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,

$$\sin^{-1}(\sin(t)) = t.$$

When solving equations involving the cosine function, we also used a negative arc identity. We do the same and will use the following negative arc identity stated on page 82.

$$\sin(-x) = -\sin(x) \text{ for every real number } x.$$

Example 2.37 (Solving an Equation Involving the Sine Function)
We will illustrate the general process using the equation $\sin(x) = -0.6$ from Progress Check 2.34. Because of the negative arc identity for the sine function,

it is actually easier to work with the equation $\sin(x) = 0.6$. This is because if $x = a$ is a solution of the equation $\sin(x) = 0.6$, then

$$\sin(-a) = -\sin(a) = -0.6,$$

and so, $x = -a$ is a solution of the equation $\sin(x) = -0.6$. For the equation $\sin(x) = 0.6$, we start by "applying the inverse sine function" to both sides of the equation.

$$\begin{aligned} \sin(x) &= 0.6 \\ \sin^{-1}(\sin(x)) &= \sin^{-1}(0.6) \\ x &= \sin^{-1}(0.6) \end{aligned}$$

We need to remember that this is only one solution of the equation. Since we know that the sine function is positive in the first and second quadrants, this solution is in the first quadrant and there is another solution in the second quadrant. Using $x = \sin^{-1}(0.6)$ as a reference arc (angle), the solution in the second quadrant is $x = \pi - \sin^{-1}(0.6)$. We now use the result that if $x = a$ is a solution of the equation $\sin(x) = 0.6$, then $x = -a$ is a solution of the equation $\sin(x) = -0.6$. Please note that

$$-(\pi - \sin^{-1}(0.6)) = -\pi + \sin^{-1}(0.6).$$

Our work so far is summarized in the following table.

Solutions for $\sin(x) = 0.6$ in $[0, \pi]$	Solutions for $\sin(x) = -0.6$ in $[-\pi, 0]$
$x = \sin^{-1}(0.6)$	$x = -\sin^{-1}(0.6)$
$x = \pi - \sin^{-1}(0.6)$	$x = -\pi + \sin^{-1}(0.6)$

At this point, we should use a calculator to verify that the two values in the right column are actually solutions of the equation $\sin(x) = -0.6$. Now that we have the solutions for $\sin(x) = -0.6$ in one complete cycle, we can use the fact that the period of the sine function is 2π and say that the solutions of the equation $\sin(x) = -0.6$ have the form

$$x = -\sin^{-1}(0.6) + k(2\pi) \quad \text{or} \quad x = (-\pi + \sin^{-1}(0.6)) + k(2\pi),$$

where k is some integer.

Progress Check 2.38 (Solving an Equation Involving the Sine Function)
Determine all solutions of the equation $2\sin(x) + 1.2 = 2.5$ in the interval $[-\pi, \pi]$. Then use the periodic property of the sine function to write formulas that can be used to generate all the solutions of this equation. **Hint**: First use algebra to rewrite the equation in the form $\sin(x) =$ "some number".

Solving More Complicated Trigonometric Equations

We have now learned to solve equations of the form $\cos(x) = q$ and $\sin(x) = q$, where q is a real number and $-1 \leq q \leq 1$. We can use our ability to solve these types of equations to help solve more complicated equations of the form $\cos(f(x)) = q$ or $\sin(f(x)) = q$ where f is some function. The idea (which is typical in mathematics) is to convert this more complicated problem to two simpler problems. The idea is to:

1. Make the substitution $t = f(x)$ to get an equation of the form $\cos(t) = q$ or $\sin(t) = q$.
2. Solve the equation in (1) for t.
3. For each solution t of the equation in (1), solve the equation $f(x) = t$ for x. This step may be easy, difficult, or perhaps impossible depending on the equation $f(x) = t$.

This process will be illustrated in the next progress check, which will be a guided investigation for solving the equation $3\cos(2x + 1) + 6 = 5$.

Progress Check 2.39 (Solving an Equation)
We will solve the equation $3\cos(2x + 1) + 6 = 5$.

1. First, use algebra to rewrite the equation in the form $\cos(2x + 1) = -\dfrac{1}{3}$. Then, make the substitution $t = 2x + 1$.
2. Determine all solutions of the equation $\cos(t) = -\dfrac{1}{3}$ with $-\pi \leq t \leq \pi$.
3. For each of these two solutions, use $t = 2x + 1$ to find corresponding solutions for x. In addition, use the substitution $t = 2x + 1$ to write $-\pi \leq 2x + 1 \leq \pi$ and solve this inequality for x. This will give all of the solutions of the equation $\cos(2x + 1) = -\dfrac{1}{3}$ in one complete cycle of the function given by $y = \cos(2x + 1)$.
4. What is the period of the function $y = \cos(2x + 1)$. Use the results in (3) and this period to write formulas that will generate all of the solutions of the equation $\cos(2x + 1) = -\dfrac{1}{3}$. These will be the solutions of the original equation $3\cos(2x + 1) + 6 = 5$.

Solving Equations Involving the Tangent Function

Solving an equation of the form $\tan(x) = q$ is very similar to solving equations of the form $\cos(x) = q$ or $\sin(x) = q$. The main differences are the tangent function has a period of π (instead of 2π), and the equation $\tan(x) = q$ has only one solution in a complete period. We, of course, use the inverse tangent function for the equation $\tan(x) = q$.

Progress Check 2.40 (Solving an Equation Involving the Tangent Function)
Use the inverse tangent function to determine one solution of the equation $4\tan(x) + 1 = 10$ in the interval $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$. Then determine a formula that can be used to generate all solutions of this equation.

Summary of Section 2.6

In this section, we studied the following important concepts and ideas:

A **trigonometric equation** is an equation that involves trigonometric functions. If we can write the trigonometric equation in the form

"some trigonometric function of x" = a number,

then we can use the following strategy to solve the equation.

- Find one solution of the equation using the appropriate inverse trigonometric function.
- Determine all solutions of the equation within one complete period of the trigonometric function. (This often involves the use of a reference arc based on the solution obtained in the first step.)
- Use the period of the function to write formulas for all of the solutions of the trigonometric equation.

Exercises for Section 2.6

1. For each of the following equations, use a graph to approximate the solutions (to three decimal places) of the equation on the indicated interval. Then use the periodic property of the trigonometric function to write formulas that can be used to approximate any solution of the given equation.

* **(a)** $\sin(x) = 0.75$ with $-\pi \leq x \leq \pi$.

(b) $\cos(x) = 0.75$ with $-\pi \leq x \leq \pi$.

(c) $\tan(x) = 0.75$ with $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

⋆ **(d)** $\sin(x) = -0.75$ with $-\pi \leq x \leq \pi$.

(e) $\cos(x) = -0.75$ with $-\pi \leq x \leq \pi$.

(f) $\tan(x) = -0.75$ with $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

⋆ **2.** For each of the equations in Exercise (1), use an inverse trigonometric function to write the exact values of all the solutions of the equation on the indicated interval. Then use the periodic property of the trigonometric function to write formulas that can be used to generate all of the solutions of the given equation.

3. For each of the following equations, use an inverse trigonometric function to write the exact values of all the solutions of the equation on the indicated interval. Then use the periodic property of the trigonometric function to write formulas that can be used to generate all of the solutions of the given equation.

⋆ **(a)** $\sin(x) + 2 = 2.4$ with $-\pi \leq x \leq \pi$.

⋆ **(b)** $5\cos(x) + 3 = 7$ with $-\pi \leq x \leq \pi$.

(c) $2\tan(x) + 4 = 10$ with $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(d) $-3.8\sin(x) + 7 = 10$ with $-\pi \leq x \leq \pi$.

(e) $8\cos(x) + 7 = 2$ with $-\pi \leq x \leq \pi$.

4. Determine the exact values of the solutions of the given equation on one complete period of the trigonometric function that is used in the equation. Then use the periodic property of the trigonometric function to write formulas that can be used to generate all of the solutions of the given equation.

⋆ **(a)** $4\sin(2x) = 3$.

(b) $4\cos(2x) = 3$.

(c) $\cos(\pi x) = 0.6$.

⋆ **(d)** $\sin\left(\pi x - \dfrac{\pi}{4}\right) = 0.2$.

(e) $\cos\left(\pi x - \dfrac{\pi}{4}\right) = 0.2$.

5. In Example 2.17 on page 2.17, we used graphical methods to find two solutions of the equation

$$35\cos\left(\frac{5\pi}{3}t\right) + 105 = 100.$$

We found that two solutions were $t \approx 0.3274$ and $t \approx 0.8726$. Rewrite this equation and then use the inverse cosine function to determine the exact values of these two solutions. Then use the period of the function $y = 35\cos\left(\frac{5\pi}{3}t\right) + 105$ to write formulas that can be used to generate all of the solutions of the given equation.

Chapter 3

Triangles and Vectors

As was stated at the start of Chapter 1, trigonometry had its origins in the study of triangles. In fact, the word *trigonometry* comes from the Greek words for triangle measurement. We will see that we can use the trigonometric functions to help determine lengths of sides of triangles or the measure of angles in triangles. As we will see in the last two sections of this chapter, triangle trigonometry is also useful in the study of vectors.

3.1 Trigonometric Functions of Angles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we define the cosine and sine as functions of angles?
- How are the trigonometric functions defined on angles using circles of any radius?

Beginning Activity

1. How do we define an angle whose measure is one radian? See the definition on page 27.

2. Draw an angle in standard position with a measure of $\dfrac{\pi}{4}$ radians. Draw an angle in standard position with a measure of $\dfrac{5\pi}{3}$ radians.

3. What is the formula for the arc length s on a circle of radius r that is intercepted by an angle with radian measure θ? See page 36. Why does this formula imply that radians are a dimensionless quantity and that a measurement in radians can be thought of as a real number?

Some Previous Results

In Section 1.2, we defined the cosine function and the sine function using the unit circle. In particular, we learned that we could define $\cos(t)$ and $\sin(t)$ for any real number where the real number t could be thought of as the length of an arc on the unit circle.

In Section 1.3, we learned that the radian measure of an angle is the length of the arc on the unit circle that is intercepted by the angle. That is,

> An angle (in standard position) of t radians will correspond to an arc of length t on the unit circle, and this allows us to think of $\cos(t)$ and $\sin(t)$ when t is the radian measure of an angle.

So when we think of $\cos(t)$ and $\sin(t)$ (and the other trigonometric functions), we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

Figure 3.1 shows an arc on the unit circle with the corresponding angle.

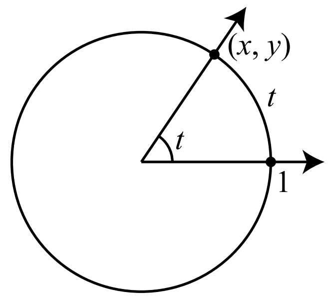


Figure 3.1: An Angle in Standard Position with the Unit Circle

Trigonometric Functions of an Angle

With the notation in Figure 3.1, we see that $\cos(t) = x$ and $\sin(t) = y$. In this context, we often call the cosine and sine *circular functions* because they are defined by points on the unit circle. Now we want to focus on the perspective of the cosine and sine as functions of angles. When using this perspective we will refer to the cosine and sine as *trigonometric functions*. Technically, we have two different types of cosines and sines: one defined as functions of arcs and the other as functions of angles. However, the connection is so close and the distinction so minor that we will often interchange the terms circular and trigonometric. One notational item is that when we think of the trigonometric functions as functions of angles, we often use Greek letters for the angles. The most common ones are θ (theta), α (alpha), β (beta), and ϕ (phi).

Although the definition of the trigonometric functions uses the unit circle, it will be quite useful to expand this idea to allow us to determine the cosine and sine of angles related to circles of any radius. The main concept we will use to do this will be similar triangles. We will use the triangles shown in Figure 3.2.

In this figure, the angle θ is in standard position, the point $P(u, v)$ is on the unit circle, and the point $Q(x, y)$ is on a circle of radius r. So we see that

$$\cos(\theta) = u \quad \text{and} \quad \sin(\theta) = v.$$

We will now use the triangles $\triangle PAO$ and $\triangle QBO$ to write $\cos(\theta)$ and $\sin(\theta)$ in terms of x, y, and r. Figure 3.3 shows these triangles by themselves without the circles.

The two triangles in Figure 3.2 are similar triangles since the corresponding angles of the two triangles are equal. (See page 426 in Appendix C.) Because of

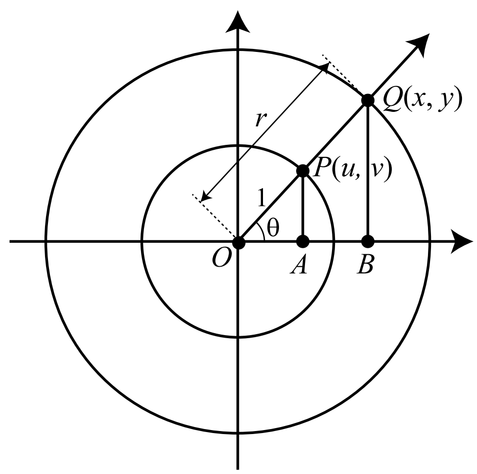


Figure 3.2: An Angle in Standard Position

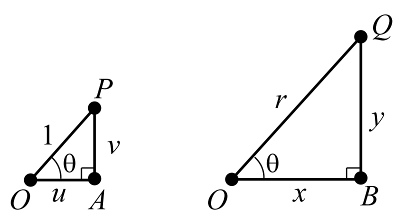


Figure 3.3: Similar Triangles from Figure 3.2

this, we can write

$$\frac{u}{1} = \frac{x}{r} \qquad \frac{v}{1} = \frac{y}{r}$$
$$u = \frac{x}{r} \qquad v = \frac{y}{r}$$
$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r}$$

In addition, note that $u^2 + v^2 = 1$ and $x^2 + y^2 = r^2$. So we have obtained the following results, which show that once we know the coordinates of one point on the terminal side of an angle θ in standard position, we can determine all six trigonometric functions of that angle.

For any point (x, y) other than the origin on the terminal side of an angle θ in standard position, the trigonometric functions of θ are defined as:

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}, x \neq 0$$

$$\sec(\theta) = \frac{r}{x}, x \neq 0 \qquad \csc(\theta) = \frac{r}{y}, y \neq 0 \qquad \cot(\theta) = \frac{x}{y}, y \neq 0$$

where $r^2 = x^2 + y^2$ and $r > 0$ and so $r = \sqrt{x^2 + y^2}$.

Notice that the other trigonometric functions can also be determined in terms of x, y, and r. For example, if $x \neq 0$, then

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{\frac{y}{r}}{\frac{x}{r}} \\ &= \frac{y}{r} \cdot \frac{r}{x} \\ &= \frac{y}{x} \end{aligned} \qquad \begin{aligned} \sec(\theta) &= \frac{1}{\cos(\theta)} \\ &= \frac{1}{\frac{x}{r}} \\ &= 1 \cdot \frac{r}{x} \\ &= \frac{r}{x} \end{aligned}$$

For example, if the point $(3, -1)$ is on the terminal side of the angle θ, then we can use $x = 3$, $y = -1$, and $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$, and so

$$\cos(\theta) = \frac{3}{\sqrt{10}} \qquad \tan(\theta) = -\frac{1}{3} \qquad \sec(\theta) = \frac{\sqrt{10}}{3}$$

$$\sin(\theta) = -\frac{1}{\sqrt{10}} \qquad \cot(\theta) = -\frac{3}{1} \qquad \csc(\theta) = -\frac{\sqrt{10}}{1}$$

The next two progress checks will provide some practice with using these results.

Progress Check 3.1 (The Trigonometric Functions for an Angle)
Suppose we know that the point $P(-3, 7)$ is on the terminal side of the angle θ in standard position.

1. Draw a coordinate system, plot the point P, and draw the terminal side of the angle θ.

2. Determine the radius r of the circle centered at the origin that passes through the point $P(-3, 7)$. **Hint**: $x^2 + y^2 = r^2$.

3. Now determine the values of the six trigonometric functions of θ.

Progress Check 3.2 (The Trigonometric Functions for an Angle)
Suppose that α is an angle, that $\tan(\alpha) = \frac{2}{3}$, and when α is in standard position, its terminal side is in the first quadrant.

1. Draw a coordinate system and draw the terminal side of the angle α in standard position.

2. Determine a point that lies on the terminal side of α.

3. Determine the six trigonometric functions of α.

The Pythagorean Identity

Perhaps the most important identity for the circular functions is the so-called Pythagorean Identity, which states that for any real number t,

$$\cos^2(t) + \sin^2(t) = 1.$$

It should not be surprising that this identity also holds for the trigonometric functions when we consider these to be functions of angles. This will be verified in the next progress check.

Progress Check 3.3 (The Pythagorean Identity)
Let θ be an angle and assume that (x, y) is a point on the terminal side of θ in standard position. We then let $r^2 = x^2 + y^2$. So we see that

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2.$$

1. Use algebra to rewrite $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$ as a single fraction with denominator r^2.

2. Now use the fact that $x^2 + y^2 = r^2$ to prove that $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$.

3. Finally, conclude that

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

The next progress check shows how to use the Pythagorean Identity to help determine the trigonometric functions of an angle.

Progress Check 3.4 (Using the Pythagorean Identity)

Assume that θ is an angle in standard position and that $\sin(\theta) = \dfrac{1}{3}$ and $\dfrac{\pi}{2} < \theta < \pi$.

1. Use the Pythagorean Identity to determine $\cos^2(\theta)$ and then use the fact that $\dfrac{\pi}{2} < \theta < \pi$ to determine $\cos(\theta)$.
2. Use the identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to determine the value of $\tan(\theta)$.
3. Determine the values of the other three trigonometric functions of θ.

The Inverse Trigonometric Functions

In Section 2.5, we studied the inverse trigonometric functions when we considered the trigonometric (circular) functions to be functions of a real number t. At the start of this section, however, we saw that t could also be considered to be the length of an arc on the unit circle, or the radian measure of an angle in standard position. At that time, we were using the unit circle to determine the radian measure of an angle but now we can use any point on the terminal side of the angle to determine the angle. The important thing is that these are now functions of angles and so we can use the inverse trigonometric functions to determine angles. We can use either radian measure or degree measure for the angles. The results we need are summarized below.

1. $\theta = \arcsin(x) = \sin^{-1}(x)$ means $\sin(\theta) = x$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$ or $-90° \le \theta \le 90°$.
2. $\theta = \arccos(x) = \cos^{-1}(x)$ means $\cos(\theta) = x$ and $0 \le \theta \le \pi$ or $0° \le \theta \le 180°$.
3. $\theta = \arctan(x) = \tan^{-1}(x)$ means $\tan(\theta) = x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ or $-90° < \theta < 90°$.

The important things to remember are that an equation involving the inverse trigonometric function can be translated to an equation involving the corresponding trigonometric function and that the angle must be in a certain range. For example, if we know that the point $(5, 3)$ is on the terminal side of an angle θ and that $0 \le \theta < \pi$, then we know that

$$\tan(\theta) = \frac{y}{x} = \frac{3}{5}.$$

We can use the inverse tangent function to determine (and approximate) the angle θ since the inverse tangent function gives an angle (in radian measure) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\tan(\theta) > 0$, we will get an angle between 0 and $\frac{\pi}{2}$. So

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 0.54042.$$

If we used degree measure, we would get

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 30.96376°.$$

It is important to note that in using the inverse trigonometric functions, we must be careful with the restrictions on the angles. For example, if we had stated that $\tan(\alpha) = \frac{5}{3}$ and $\pi < \alpha < \frac{3\pi}{2}$, then the inverse tangent function would not give the correct result. We could still use

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 0.54042,$$

but now we would have to use this result and the fact that the terminal side of α is in the third quadrant. So

$$\begin{aligned}
\alpha &= \theta + \pi \\
\alpha &= \arctan\left(\frac{3}{5}\right) + \pi \\
\alpha &\approx 3.68201
\end{aligned}$$

We should now use a calculator to verify that $\tan(\alpha) = \frac{3}{5}$.

The relationship between the angles α and θ is shown in Figure 3.4.

Progress Check 3.5 (Finding an Angle)

Suppose that the point $(-2, 5)$ is on the terminal side of the angle θ in standard position and that $0 \le \theta < 360°$. We then know that $\tan(\theta) = -\frac{5}{2} = -2.5$.

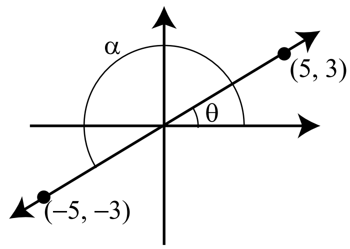


Figure 3.4: Two Angles with the Same Tangent Value

1. Draw a picture of the angle θ.

2. Use a calculator to approximate the value of $\tan^{-1}(-2.5)$ to three decimal places.

3. Notice that $\tan^{-1}(-2.5)$ is a negative angle and cannot equal θ since θ is a positive angle. Use the approximation for $\tan^{-1}(-2.5)$ to determine an approximation for θ to three decimal places.

In the following example, we will determine the exact value of an angle that is given in terms of an inverse trigonometric function.

Example 3.6 Determining an Exact Value

We will determine the exact value of $\cos\left(\arcsin\left(-\dfrac{2}{7}\right)\right)$. Notice that we can use a calculator to determine that

$$\cos\left(\arcsin\left(-\frac{2}{7}\right)\right) \approx 0.958315.$$

Even though this is correct to six decimal places, it is not the exact value. We can use this approximation, however, to check our work below.

We let $\theta = \arcsin\left(-\dfrac{2}{7}\right)$. We then know that

$$\sin(\theta) = -\frac{2}{7} \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We note that since $\sin(\theta) < 0$, we actually know that $-\dfrac{\pi}{2} \leq \theta \leq 0$.

So we can use the Pythagorean Identity to determine $\cos^2(\theta)$ as follows:

$$\begin{aligned}\cos^2(\theta) + \sin^2(\theta) &= 1 \\ \cos^2(\theta) &= 1 - \left(-\frac{2}{7}\right)^2 \\ \cos^2(\theta) &= \frac{45}{49}\end{aligned}$$

Since $-\frac{\pi}{2} \le \theta \le 0$, we see that $\cos(\theta) = \frac{\sqrt{45}}{7}$. That is

$$\cos\left(\arcsin\left(-\frac{2}{7}\right)\right) = \frac{\sqrt{45}}{7}.$$

We can now use a calculator to verify that $\frac{\sqrt{45}}{7} \approx 0.958315$.

Summary of Section 3.1

In this section, we studied the following important concepts and ideas:

The trigonometric functions can be defined using any point on the terminal side of an angle in standard position. For any point (x, y) other than the origin on the terminal side of an angle θ in standard position, the trigonometric functions of θ are defined as:

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}, x \ne 0$$

$$\sec(\theta) = \frac{r}{x}, x \ne 0 \qquad \csc(\theta) = \frac{r}{y}, y \ne 0 \qquad \cot(\theta) = \frac{x}{y}, y \ne 0$$

where $r^2 = x^2 + y^2$ and $r > 0$ and so $r = \sqrt{x^2 + y^2}$. The Pythagorean Identity is still true when we use the trigonometric functions of an angle. That is, for any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

In addition, we still have the inverse trigonometric functions. In particular,

- $\theta = \arcsin(x) = \sin^{-1}(x)$ means $\sin(\theta) = x$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $-90° \le \theta \le 90°$.
- $\theta = \arccos(x) = \cos^{-1}(x)$ means $\cos(\theta) = x$ and $0 \le \theta \le \pi$ or $0° \le \theta \le 180°$.

- $\theta = \arctan(x) = \tan^{-1}(x)$ means $\tan(\theta) = x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ or $-90° < \theta < 90°$.

Exercises for Section 3.1

1. In each of the following, the coordinates of a point P on the terminal side of an angle θ are given. For each of the following:

- Plot the point P in a coordinate system and draw the terminal side of the angle.
- Determine the radius r of the circle centered at the origin that passes through the point P.
- Determine the values of the six trigonometric functions of the angle θ.

⋆ **(a)** $P(3, 3)$	**(d)** $P(5, -2)$	**(g)** $P(-3, 4)$
⋆ **(b)** $P(5, 8)$	⋆ **(e)** $P(-1, -4)$	**(h)** $P(3, -3\sqrt{3})$
(c) $P(-2, -2)$	**(f)** $P(2\sqrt{3}, 2)$	**(i)** $P(2, -1)$

2. For each of the following, draw the terminal side of the indicated angle on a coordinate system and determine the values of the six trigonometric functions of that angle

(a) The terminal side of the angle α is in the first quadrant and $\sin(\alpha) = \dfrac{1}{\sqrt{3}}$.

⋆ **(b)** The terminal side of the angle β is in the second quadrant and $\cos(\beta) = -\dfrac{2}{3}$.

(c) The terminal side of the angle γ is in the second quadrant and $\tan(\gamma) = -\dfrac{1}{2}$.

(d) The terminal side of the angle θ is in the second quadrant and $\sin(\theta) = \dfrac{1}{3}$.

3. For each of the following, determine an approximation for the angle θ in degrees (to three decimal places) when $0° \leq \theta < 360°$.

(a) The point $(3, 5)$ is on the terminal side θ.

(**b**) The point $(2, -4)$ is on the terminal side of θ.

⋆ (**c**) $\sin(\theta) = \dfrac{2}{3}$ and the terminal side of θ is in the second quadrant.

(**d**) $\sin(\theta) = -\dfrac{2}{3}$ and the terminal side of θ is in the fourth quadrant.

⋆ (**e**) $\cos(\theta) = -\dfrac{1}{4}$ and the terminal side of θ is in the second quadrant.

(**f**) $\cos(\theta) = -\dfrac{3}{4}$ and the terminal side of θ is in the third quadrant.

4. For each of the angles in Exercise (3), determine the radian measure of θ if $0 \le \theta < 2\pi$.

5. Determine the exact value of each of the following. Check all results with a calculator.

(**a**) $\cos\left(\arcsin\left(\dfrac{1}{5}\right)\right)$.

⋆ (**b**) $\tan\left(\cos^{-1}\left(\dfrac{2}{3}\right)\right)$.

(**c**) $\sin\left(\tan^{-1}(2)\right)$.

(**d**) $\cos\left(\arcsin\left(-\dfrac{1}{5}\right)\right)$.

(**e**) $\sin\left(\arccos\left(-\dfrac{3}{5}\right)\right)$.

3.2 Right Triangles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How does the cosine relate sides and acute angles in a right triangle? Why?
- How does the sine relate sides and acute angles in a right triangle? Why?
- How does the tangent relate sides and acute angles in a right triangle? Why?
- How can we use the cosine, sine, and tangent of an angle in a right triangle to help determine unknown parts of that triangle?

Beginning Activity

Figure 3.5 shows a typical right triangle. The lengths of the three sides of the right triangle are labeled as a, b, and c. The angles opposite the sides of lengths a, b, and c are labeled α (alpha), β (beta), and γ (gamma), respectively. (Alpha, beta, and gamma are the first three letters in the Greek alphabet.) The small square with the angle γ indicates that this is the right angle in the right triangle. The triangle, of course, has three sides. We call the side opposite the right angle (the side of length c in the diagram) the **hypotenuse** of the right triangle.

When we work with triangles, the angles are usually measured in degrees and so we would say that γ is an angle of 90°.

1. What can we conclude about a, b, and c from the Pythagorean Theorem?

When working with triangles, we usually measure angles in degrees. For the fractional part of the degree measure of an angle, we often used decimals but we also frequently use minutes and seconds.

2. What is the sum of the angles in a triangle? In this case, what is $\alpha + \beta + \gamma$?

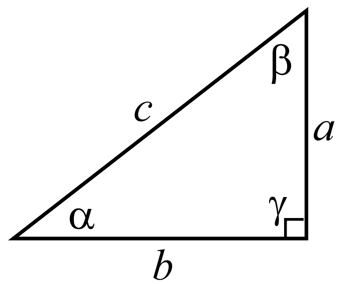


Figure 3.5: A typical right triangle

3. What is the sum of the two acute angles in a right triangle. In this case, what is $\alpha + \beta$?

4. How many minutes are in a degree? How many seconds are in a minute?

5. Determine the solution of the equation $7.3 = \dfrac{118.8}{x}$ correct to the nearest thousandth. (You should be able to show that $x \approx 16.274$.)

6. Determine the solution of the equation $\sin(32°) = \dfrac{5}{x}$ correct to the nearest ten-thousandth. (You should be able to show that $x \approx 9.4354$.)

Introduction

Suppose you want to find the height of a tall object such as a flagpole (or a tree or a building). It might be inconvenient (or even dangerous) to climb the flagpole and measure it, so what can you do? It might be easy to measure the length the shadow the flagpole casts and also the angle θ determined by the ground level to the sun (called the **angle of elevation of the object**) as in Figure 3.6. In this section, we will learn how to use the trigonometric functions to relate lengths of sides to angles in right triangles and solve this problem as well as many others.

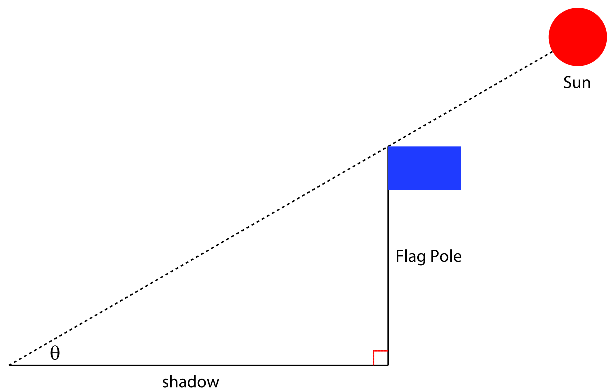


Figure 3.6: Finding the height of a flagpole (drawing not to scale)

Trigonometric Functions and Right Triangles

We have seen how we determine the values of the trigonometric functions of an angle θ by placing θ in standard position and letting (x, y) be the point of intersection of the terminal side of angle θ with a circle of radius r. Then

$$\cos(\theta) = \frac{x}{r}, \qquad \sec(\theta) = \frac{r}{x} \text{ if } x \neq 0,$$
$$\sin(\theta) = \frac{y}{r}, \qquad \csc(\theta) = \frac{r}{y} \text{ if } y \neq 0,$$
$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0, \qquad \cot(\theta) = \frac{x}{y} \text{ if } y \neq 0.$$

In our work with right triangles, we will use only the sine, cosine, and tangent functions.

Now we will see how to relate the trigonometric functions to angles in right triangles. Suppose we have a right triangle with sides of length x and y and hypotenuse of length r. Let θ be the angle opposite the side of length y as shown in Figure 3.7. We can now place our triangle such that the angle θ is in standard position in the plane and the triangle will fit into the circle of radius r as shown at right in Figure 3.8. By the definition of our trigonometric functions we then have

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}$$

If instead of using x, y, and r, we label y as the length of the side opposite the acute angle θ, x as the length of the side adjacent to the acute angle θ, and r as the length of the hypotenuse, we get Figure 3.9.

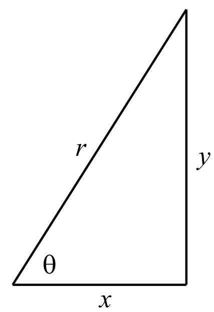


Figure 3.7: A right triangle

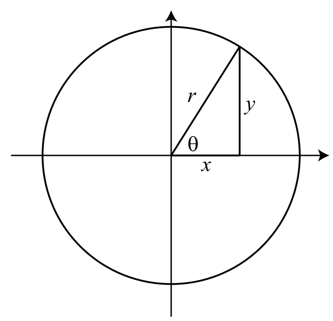


Figure 3.8: Right triangle in standard position

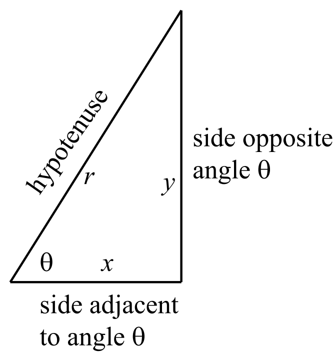


Figure 3.9: A right triangle

So we see that

$$\sin(\theta) = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

The equations on the right are convenient abbreviations of the correct equations on the left.

Progress Check 3.7 (Labeling a Right Triangle)
We must be careful when we use the terms opposite and adjacent because the meaning of these terms depends on the angle we are using. Use the diagrams in Figure 3.10 to determine formulas for each of the following in terms of a, b, and c.

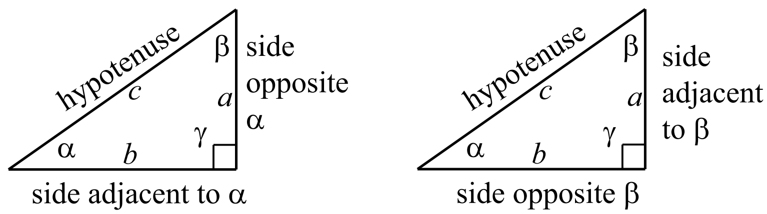


Figure 3.10: Labels for a right triangle

$\cos(\alpha) =$ __________ $\cos(\beta) =$ __________

$\sin(\alpha) =$ __________ $\sin(\beta) =$ __________

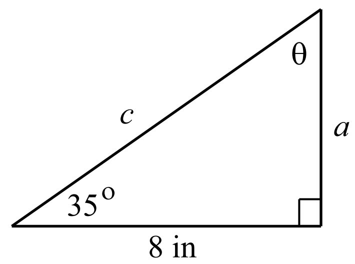
$\tan(\alpha) =$ __________ $\tan(\beta) =$ __________

We should also note that with the labeling of the right triangle shown in Figure 3.10, we can use the Pythagorean Theorem and the fact that the sum of the angles of a triangle is 180 degrees to conclude that

$$a^2 + b^2 = c^2 \quad \text{and} \quad \begin{aligned} \alpha + \beta + \gamma &= 180^\circ \\ \gamma &= 90^\circ \\ \alpha + \beta &= 90^\circ \end{aligned}$$

Example 3.8 Suppose that one of the acute angles of a right triangle has a measure of 35° and that the side adjacent to this angle is 8 inches long. Determine the other acute angle of the right triangle and the lengths of the other two sides.

Solution. The first thing we do is draw a picture of the triangle. (The picture does not have to be perfect but it should reasonably reflect the given information.) In making the diagram, we should also label the unknown parts of the triangle. One way to do this is shown in the diagram.



One thing we notice is that $35° + \theta = 90°$ and so $\theta = 55°$. We can also use the cosine and tangent of 35° to determine the values of a and c.

$$\begin{aligned} \cos(35°) &= \frac{8}{c} \\ c\cos(35°) &= 8 \\ c &= \frac{8}{\cos(35°)} \\ c &\approx 9.76620 \end{aligned} \qquad \begin{aligned} \tan(35°) &= \frac{a}{8} \\ 8\tan(35°) &= a \\ a &\approx 5.60166 \end{aligned}$$

Before saying that this example is complete, we should check our results. One way to do this is to verify that the lengths of the three sides of the right triangle satisfy the formula for the Pythagorean Theorem. Using the given value for one side and the calculated values of a and c, we see that

$$\begin{aligned} 8^2 + a^2 &\approx 95.379 \\ c^2 &\approx 95.379 \end{aligned}$$

So we see that our work checks with the Pythagorean Theorem.

Solving Right Triangles

What we did in Example 3.8 is what is called **solving a right triangle**. Please note that this phrase is misleading because you cannot really "solve" a triangle. However, since this phrase is a traditional part of the vernacular of trigonometry and so we will continue to use it. The idea is that if we are given enough information about the lengths of sides and measures of angles in a right triangle, then we can determine all of the other values. The next progress check is also an example of "solving a right triangle."

Progress Check 3.9 (Solving a Right Triangle)
The length of the hypotenuse of a right triangle is 17 feet and the length of one side of this right triangle is 5 feet. Determine the length of the other side and the two acute angles for this right triangle.

Hint: Draw a picture and label the third side of the right triangle with a variable and label the two acute angles as α and β.

Applications of Right Triangles

As the examples have illustrated up to this point, when working on problems involving right triangles (including application problems), we should:

- Draw a diagram for the problem.
- Identify the things you know about the situation. If appropriate, include this information in your diagram.
- Identify the quantity that needs to be determined and give this quantity a variable name. If appropriate, include this information in your diagram.
- Find an equation that relates what is known to what must be determined. This will often involve a trigonometric function or the Pythagorean Theorem.
- Solve the equation for the unknown. Then think about this solution to make sure it makes sense in the context of the problem.
- If possible, find a way to check the result.

We return to the example given in the introduction to this section on page 179. In this example, we used the term *angle of elevation*. This is a common term (as well as *angle of depression*) in problems involving triangles. We can define an **angle of elevation** of an object to be an angle whose initial side is horizontal and has a rotation so that the terminal side is above the horizontal. An **angle of depression** is then an angle whose initial side is horizontal and has a rotation so that the terminal side is below the horizontal. See Figure 3.11.

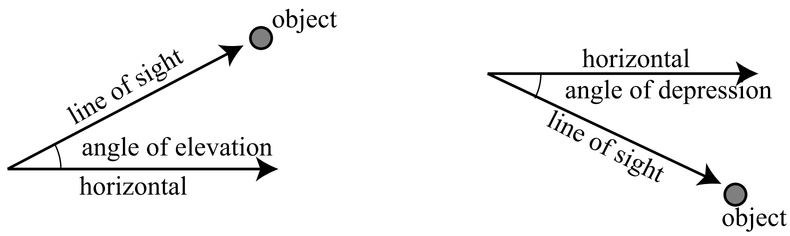


Figure 3.11: Angle of Elevation and Angle of Depression

Example 3.10 Determining the Height of a Flagpole
Suppose that we want to determine the height of a flagpole and cannot measure the height directly. Suppose that we measure the length of the shadow of the flagpole to be 44 feet, 5 inches. In addition, we measure the angle of elevation of the sun to be $33°15'$.

Solution. The first thing we do is to draw the diagram. In the diagram, we let h

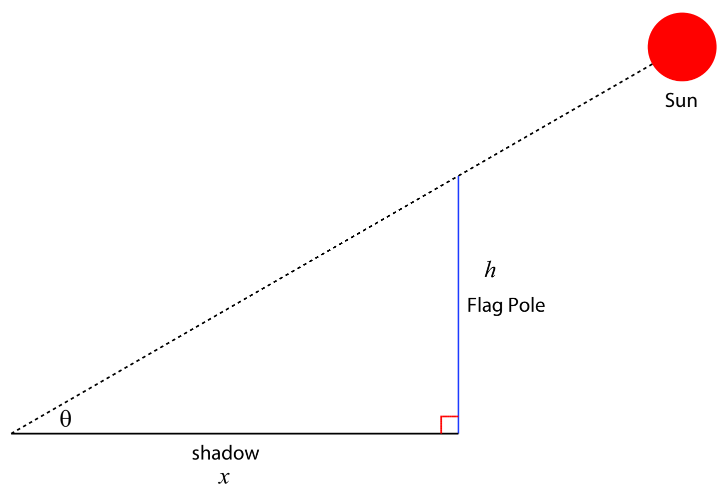


Figure 3.12: Finding the height of a flagpole (drawing not to scale)

be the height of the flagpole, x be the length of the shadow, and θ be the angle of elevation. We are given values for x and θ, and we see that

$$\begin{aligned} \tan(\theta) &= \frac{h}{x} \\ x\tan(\theta) &= h \qquad (1) \end{aligned}$$

So we can now determine the value of h, but we must be careful to use a decimal (or fractional) value for x (equivalent to 44 feet, 5 inches) and a decimal (or fractional) value for θ (equivalent to $33^\circ 15'$). So we will use

$$x = 44 + \frac{5}{12} \quad \text{and} \quad \theta = \left(33 + \frac{15}{60}\right)^\circ.$$

Using this and equation (1), we see that

$$\begin{aligned} h &= \left(44 + \frac{5}{12}\right)\tan\left(33 + \frac{15}{60}\right)^\circ \\ h &\approx 29.1208 \text{ feet.} \end{aligned}$$

The height of the flagpole is about 29.12 feet or 29 feet, 1.4 inches.

Progress Check 3.11 (Length of a Ramp)
A company needs to build a wheelchair accessible ramp to its entrance. The Americans with Disabilities Act Guidelines for Buildings and Facilities for ramps state the "The maximum slope of a ramp in new construction shall be 1:12."

1. The 1:12 guideline means that for every 1 foot of rise in the ramp there must be 12 feet of run. What is the angle of elevation (in degrees) of such a ramp?

2. If the company's entrance is 7.5 feet above the level ground, use trigonometry to approximate the length of the ramp that the company will need to build using the maximum slope. Explain your process.

Progress Check 3.12 (Guided Activity – Using Two Right Triangles)
This is a variation of Example 3.19. Suppose that the flagpole sits on top a hill and that we cannot directly measure the length of the shadow of the flagpole as shown in Figure 3.19.

Some quantities have been labeled in the diagram. Angles α and β are angles of elevation to the top of the flagpole from two different points on level ground. These points are d feet apart and directly in line with the flagpole. The problem

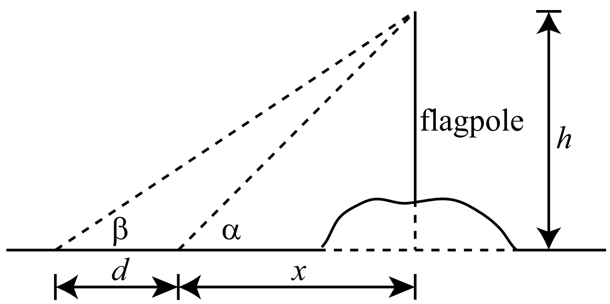


Figure 3.13: Flagpole on a hill

is to determine h, the height from level ground to the top of the flagpole. The following measurements have been recorded.

$$\alpha = 43.2^\circ \qquad d = 22.75\text{feet}$$
$$\beta = 34.7^\circ$$

Notice that a value for x was not given because it is the distance from the first point to an imaginary point directly below the flagpole and even with level ground.

Please keep in mind that it is probably easier to write formulas in terms of α, β, and γ and wait until the end to use the numerical values. For example, we see that

$$\tan(\alpha) = \frac{h}{x} \quad \text{and} \tag{1}$$
$$\tan(\beta) = \frac{h}{d+x}. \tag{2}$$

In equation (1), notice that we know the value of α. This means if we can determine a value for either x or h, we can use equation (1) to determine the value of the other. We will first determine the value of x.

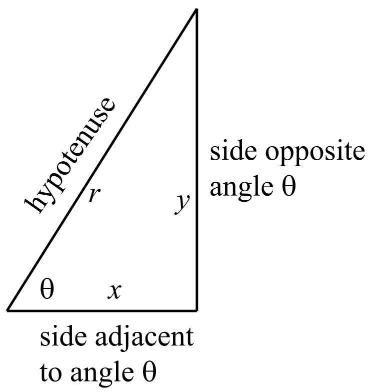
1. Solve equation (1) for h and then substitute this into equation (2). Call this equation (3).
2. One of the terms in equation (3) has a denominator. Multiply both sides of equation (3) by this denominator.
3. Now solve the resulting equation for x (in terms of α, β, and d).

4. Substitute the given values for α, β, and d to determine the value of x and then use this value and equation (1) to determine the value of h.

5. Is there a way to check to make sure the result is correct?

Summary of Section 3.2

In this section, we studied the following important concepts and ideas:
Given enough information about the lengths of sides and measures of angles in a right triangle, we can determine all of the other values using the following relationships:



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

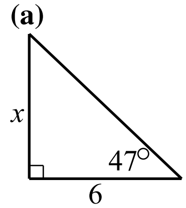
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$x^2 + y^2 = r^2$$

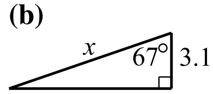
Exercises for Section 3.2

⋆ **1.** For each of the following right triangles, determine the value of x correct to the nearest thousandth.

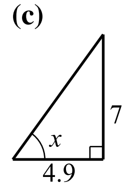
(a)



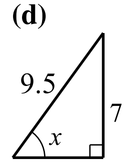
(b)



(c)



(d)



2. One angle in a right triangle is 55° and the side opposite that angle is 10 feet long. Determine the length of the other side, the length of the hypotenuse, and the measure of the other acute angle.

3. One angle in a right triangle is 37.8° and the length of the hypotenuse is 25 inches. Determine the length of the other two sides of the right triangle.

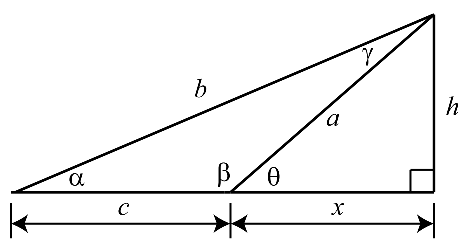
⋆ **4.** One angle in a right triangle is $27°12'$ and the length of the side adjacent to this angle is 4 feet. Determine the other acute angle in the triangle, the length of the side opposite this angle, and the length of the hypotenuse.
Note: The notation means that the angle is 27 degrees, 12 seconds. Recall that 1 second is $\dfrac{1}{60}$ of a degree.

5. If we only know the measures of the three angles of a right triangle, explain why it is not possible to determine the lengths of the sides of this right triangle.

6. Suppose that we know the measure θ of one of the acute angles in a right triangle and we know the length x of the side opposite the angle θ. Explain how to determine the length of the side adjacent to the angle θ and the length of the hypotenuse.

⋆ **7.** In the diagram to the right, determine the values of a, b, and h to the nearest thousandth.
The given values are:

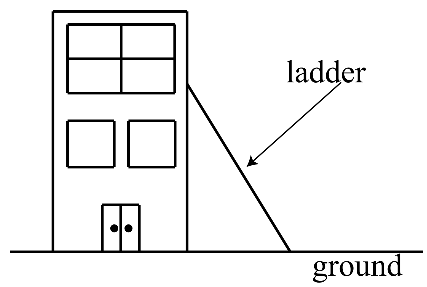
$$\alpha = 23°$$
$$\beta = 140°$$
$$c = 8$$



8. A tall evergreen tree has been damaged in a strong wind. The top of the tree is cracked and bent over, touching the ground as if the trunk were hinged. The tip of the tree touches the ground 20 feet 6 inches from the base of the tree (where the tree and the ground meet). The tip of the tree forms an angle of 17 degrees where it touches the ground. Determine the original height of the tree (before it broke) to the nearest tenth of a foot. Assume the base of the tree is perpendicular to the ground.

9. Suppose a person is standing on the top of a building and that she has an instrument that allows her to measure angles of depression. There are two points that are 100 feet apart and lie on a straight line that is perpendicular to the base of the building. Now suppose that she measures the angle of depression to the closest point to be 35.5° and that she measures the angle of depression to the other point to be 29.8°. Determine the height of the building.

10. A company has a 35 foot ladder that it uses for cleaning the windows in their building. For safety reasons, the ladder must never make an angle of more than 50° with the ground.



(**a**) What is the greatest height that the ladder can reach on the building if the angle it makes with the ground is no more than 50°.

(**b**) Suppose the building is 40 feet high. Again, following the safety guidelines, what length of ladder is needed in order to have the ladder reach the top of the building?

3.3 Triangles that Are Not Right Triangles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the Law of Sines?
- What information do we need about a triangle to apply the Law of Sines?
- What do we mean by the ambiguous case for the Law of Sines? Why is it ambiguous?
- What is the Law of Cosines?
- What information do we need about a triangle to apply the Law of Cosines?

Introduction

In Section 3.2, we learned how to use the trigonometric functions and given information about a right triangle to determine other parts of that right triangle. Of course, there are many triangles without right angles (these triangles are called *oblique triangles*). Our next task is to develop methods to relate sides and angles of oblique triangles. In this section, we will develop two such methods, the Law of Sines and the Law of Cosines. In the next section, we will learn how to use these methods in applications.

As with right triangles, we will want some standard notation when working with general triangles. Our notation will be similar to the what we used for right triangles. In particular, we will often let the lengths of the three sides of a triangle be a, b, and c. The angles opposite the sides of length a, b, and c will be labeled α, β, and γ, respectively. See Figure 3.14.

We will sometimes label the vertices of the triangle as A, B, and C as shown in Figure 3.14.

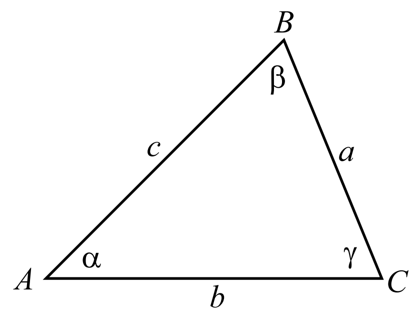


Figure 3.14: Standard Labeling for a Triangle

Beginning Activity

Before we state the Law of Sines and the Law of Cosines, we are going to use two Geogebra apps to explore the relationships about the parts of a triangle. In each of these apps, a triangle is drawn. The lengths of the sides of the triangle and the measure for each of the angles in shown. The size and shape of the triangle can be changed by dragging one (or all) of the points that form the vertices of the triangle.

1. Open the Geogebra app called *The Law of Sines* at

http://gvsu.edu/s/01B

(**a**) Experiment by moving the vertices of the triangle and observing what happens with the lengths and the angles and the computations shown in the lower left part of the screen.

(**b**) Use a particular triangle and verify the computations shown in the lower left part of the screen. Round your results to the nearest thousandth as is done in the app.

(**c**) Write an equation (or equations) that this app is illustrating. This will be part of the Law of Sines.

2. Open the Geogebra app called *The Law of Cosines* at

http://gvsu.edu/s/01C

(**a**) Experiment by moving the vertices of the triangle and observing what happens with the lengths and the angles and the computations shown in the lower left part of the screen.

(b) Use a particular triangle and verify the computations shown in the lower left part of the screen. Round your results to the nearest thousandth as is done in the app.

(c) Write an equation that this app is illustrating. This will be part of the Law of Cosines.

The Law of Sines

The first part of the beginning activity was meant to illustrate the Law of Sines. Following is a formal statement of the Law of Sines.

Law of Sines
In a triangle, if a, b, and c are the lengths of the sides opposite angles α, β, and γ, respectively, then

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

This is equivalent to

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}.$$

A proof of the Law of Sines is included at the end of this section.

Please note that the Law of Sines actually has three equations condensed into a single line. The three equations are:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} \qquad \frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c} \qquad \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

The key to using the Law of Sines is that each equation involves 4 quantities, and if we know 3 of these quantities, we can use the Law of Sines to determine the fourth. These 4 quantities are actually two different pairs, where one element of a pair is an angle and the other element of that pair is the length of the side opposite that angle. In Figure 3.15, θ and x form one such pair, and ϕ and y are another such pair. We can write the Law of Sines as follows:

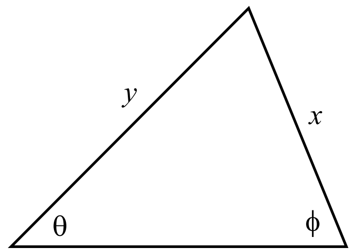


Figure 3.15: Diagram for the Law of Sines

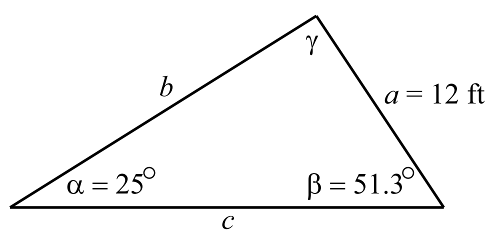
Law of Sines

In a triangle, if x is the length of the side opposite angle θ and y is the length of the side opposite angle ϕ, then

$$\frac{x}{\sin(\theta)} = \frac{y}{\sin(\phi)} \quad \text{or} \quad \frac{\sin(\theta)}{x} = \frac{\sin(\phi)}{y}.$$

Example 3.13 (Using the Law of Sines)

Suppose that the measures of two angles of a triangle are 25° and 51.3° and that the side opposite the 25° angle is 12 feet long. We will use the Law of Sines to determine the other three parts of the triangle. (Remember that we often say that we are "solving the triangle.") The first step is to draw a reasonably accurate diagram of the triangle and label the parts. This is shown in the following diagram.



We notice that we know the values of the length of a side and its opposite angles (a and α). Since we also know the value of β, we can use the Law of Sines to determine b. This is done as follows:

$$\begin{aligned}\frac{a}{\sin(\alpha)} &= \frac{b}{\sin(\beta)}\\ b &= \frac{a\sin(\beta)}{\sin(\alpha)}\\ b &= \frac{12\sin(51.3°)}{\sin(25°)}\\ b &\approx 22.160\end{aligned}$$

So we see that the side opposite the $51.3°$ angle is about 22.160 feet in length. We still need to determine γ and c. We will use the fact that the sum of the angles of a triangle is equal to $180°$ to determine γ.

$$\begin{aligned}\alpha + \beta + \gamma &= 180°\\ 25° + 51.3° + \gamma &= 180°\\ \gamma &= 103.7°\end{aligned}$$

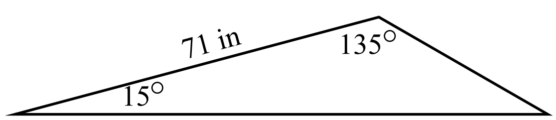
Now that we know γ, we can use the Law of Sines again to determine c. To do this, we solve the following equation for c.

$$\frac{a}{\sin(\alpha)} = \frac{c}{\sin(\gamma)}.$$

We should verify that the result is $c \approx 27.587$ feet. To check our results, we should verify that for this triangle,

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \approx 0.035.$$

Progress Check 3.14 (Using the Law of Sines)
Suppose that the measures of two angles of a triangle are $15°$ and $135°$ and that the side that is common to these two angles is 71 inches long. Following is a reasonably accurate diagram for this triangle.



Determine the lengths of the other two sides of the triangle and the measure of the third angle. **Hint**: First introduce some appropriate notation, determine the measure of the third angle, and then use the Law of Sines.

Using the Law of Sines to Determine an Angle

As we have stated, an equation for the Law of Sines involves four quantities, two angles and the lengths of the two sides opposite these angles. In the examples we have looked at, two angles and one side has been given. We then used the Law of Sines to determine the length of the other side.

We can run into a slight complication when we want to determine an angle using the Law of Sines. This can occur when we are given the lengths of two sides and the measure of an angle opposite one of these sides. The problem is that there are two different angles between 0° and 180° that are solutions of an equation of the form

$$\sin(\theta) = \text{“a number between 0 and 1”}.$$

For example, consider the equation $\sin(\theta) = 0.7$. We can use the inverse sine function to determine one solution of this equation, which is

$$\theta_1 = \sin^{-1}(0.7) \approx 44.427°.$$

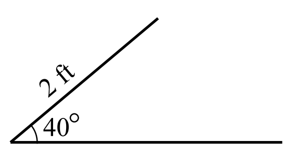
The inverse sine function gives us the solution that is between 0° and 90°, that is, the solution in the first quadrant. There is a second solution to this equation in the second quadrant, that is, between 90° and 180°. This second solution is $\theta_2 = 180° - \theta_1$. So in this case,

$$\theta_2 = 180° - \sin^{-1}(0.7) \approx 135.573°.$$

The next two progress checks will be guided activities through examples where we will need to use the Law of Sines to determine an angle.

Progress Check 3.15 (Using the Law of Sines for an Angle)
Suppose a triangle has a side of length 2 feet that is an adjacent side for an angle of 40°. Is it possible for the side opposite the 40° angle to have a length of 1.7 feet?

To try to answer this, we first draw a reasonably accurate diagram of the situation as shown below.



The horizontal line is not a side of the triangle (yet). For now, we are just using it as one of the sides of the 40° angle. In addition, we have not drawn the side opposite the 40° angle since just by observation, it appears there could be two possible ways to draw a side of length 1.7 feet. Now we get to the details.

1. Let θ be the angle opposite the side of length 2 feet. Use the Law of Sines to determine $\sin(\theta)$.

2. Use the inverse sine function to determine one solution (rounded to the nearest tenth of a degree) for θ. Call this solution θ_1.

3. Let $\theta_2 = 180° - \theta_1$. Explain why (or verify that) θ_2 is also a solution of the equation in part (1).

This means that there could be two triangles that satisfy the conditions of the problem.

4. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_1.

5. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_2.

There are times when the Law of Sines will show that there are no triangles that meet certain conditions. We often see this when an equation from the Law of Sines produces an equation of the form

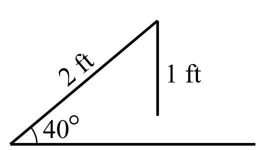
$$\sin(\theta) = p,$$

where p is real number but is not between 0 and 1. For example, changing the conditions in Progress Check 3.15 so that we want a triangle that has a side of length 2 feet that is an adjacent side for an angle of 40° and the side opposite the 40° angle is to have a length of 1 foot. As in Progress Check 3.15, we let θ be the angle opposite the side of length 2 feet and use the Law of Sines to obtain

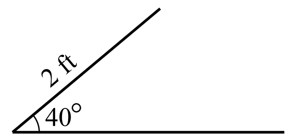
$$\frac{\sin(\theta)}{2} = \frac{\sin(40°)}{1}$$

$$\sin(\theta) = \frac{2\sin(40°)}{1} \approx 1.2856$$

There is no such angle θ and this shows that there is no triangle that meets the specified conditions. The diagram on the right illustrates the situation.



Progress Check 3.16 (Using the Law of Sines for an Angle)
Suppose a triangle has a side of length 2 feet that is an adjacent side for an angle of 40°. Is it possible for the side opposite the 40° angle to have a length of 3 feet?



The only difference between this and Progress Check 3.15 is in the length of the side opposite the 40° angle. We can use the same diagram. By observation, it appears there is likely only way to draw a side of length 3 feet. Now we get to the details.

1. Let θ be the angle opposite the side of length 2 feet. Use the Law of Sines to determine $\sin(\theta)$.

2. Use the inverse sine function to determine one solution (rounded to the nearest tenth of a degree) for θ. Call this solution θ_1.

3. Let $\theta_2 = 180° - \theta_1$. Explain why (or verify that) θ_2 is also a solution of the equation in part (1).

This means that there could be two triangles that satisfy the conditions of the problem.

4. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_1.

5. Now determine the sum $40° + \theta_2$ and explain why this is not possible in a triangle.

Law of Cosines

We have seen how the Law of Sines can be used to determine information about sides and angles in oblique triangles. However, to use the Law of Since we need to know three pieces of information. We need to know an angle and the length of its opposite side, and in addition, we need to know another angle or the length of another side. If we have three different pieces of information such as the lengths of two sides and the included angle between them or the lengths of the three sides, then we need a different method to determine the other pieces of information about the triangle. This is where the Law of Cosines is useful.

We first explored the Law of Cosines in the beginning activity for this section. Following is the usual formal statement of the Law of Cosines. A proof of the Law of Cosines is included at the end of this section.

Law of Cosines
In a triangle, if a, b, and c are the lengths of the sides opposite angles α, β, and γ, respectively, then

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$
$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

As with the Law of Sines, there are three equations in the Law of Cosines. However, we can remember this with only one equation since the key to using the Law of Cosines is that this law involves 4 quantities. These 4 quantities are the lengths of the three sides and the measure of one of the angles of the triangle as shown in Figure 3.16.

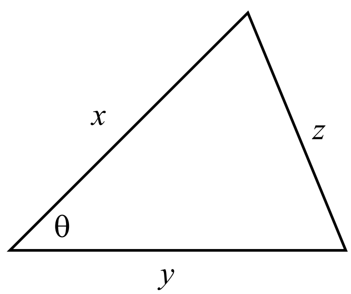


Figure 3.16: Diagram for the Law of Cosines

In this diagram, x, y, and z are the lengths of the three sides and θ is the angle between the sides x and y. Theta can also be thought of as the angle opposite side z. So we can write the Law of Cosines as follows:

Law of Cosines
In a triangle, if x, y, and z are the lengths of the sides of a triangle and θ is the angle between the sides x and y as in Figure 3.16, then

$$z^2 = x^2 + y^2 - 2xy\cos(\theta).$$

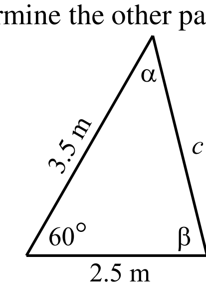
The idea is that if you know 3 of these 4 quantities, you can use the Law of Cosines to determine the fourth quantity. The Law of Cosines involves the lengths of all three sides of a triangle and one angle. It states that:

The square of the side opposite an angle is the sum of the squares of the two sides of the angle minus two times the product of the two sides of the angle and the cosine of the angle.

We will explore the use of the Law of Cosines in the next progress check.

Progress Check 3.17 (Using the Law of Cosines
Two sides of a triangle have length 2.5 meters and 3.5 meters, and the angle formed by these two sides has a measure of 60°. Determine the other parts of the triangle.

The first step is to draw a reasonably accurate diagram of the triangle and label the parts. This is shown in the diagram on the right.



1. Use the Law of Cosines to determine the length of the side opposite the 60° angle. (c).

We now know an angle (60°) and the length of its opposite side. We can use the Law of Sines to determine the other two angles. However, remember that we must be careful when using the Law of Sines to determine an angle since the equation may produce two angles.

2. Use the Law of Sines to determine $\sin(\alpha)$. Determine the two possible values for α and explain why one of them is not possible.

3. Use the fact that the sum of the angles of a triangle is 180° to determine the angle β.

4. Use the Law of Sines to check the results.

We used the Law of Sines to determine two angles in Progress Check 3.17 and saw that we had to be careful since the equation for the Law of Sines often produces two possible angles. We can avoid this situation by using the Law of Cosines to determine the angles instead. This is because an equation of the form $\cos(\theta) = p$, where p is a real number between 0 and 1 has only one solution for θ between $0°$ and $180°$. The idea is to solve an equation from the Law of Cosines for the cosine of the angle. In Progress Check 3.17, we first determined $c^2 = 9.75$ or $c \approx 3.12250$.. We then could have proceeded as follows:

$$\begin{aligned} 2.5^2 &= 3.5^2 + 3.12250^2 - 2(3.5)(3.12250)\cos(\alpha) \\ 2(3.5)(3.12250)\cos(\alpha) &= 3.5^2 + 3.12250^2 - 2.5^2 \\ \cos(\alpha) &= \frac{15.75}{21.8575} \approx 0.720577 \end{aligned}$$

We can then use the inverse cosine function and obtain $\alpha \approx 43.898°$, which is what we obtained in Progress Check 3.17.

We can now use the fact that the sum of the angles in a triangle is $180°$ to determine β but for completeness, we could also use the Law of Cosines to determine β and then use the angle sum for the triangle as a check on our work.

Progress Check 3.18 (Using the Law of Cosines)
The three sides of a triangle have lengths of 3 feet, 5 feet, and 6 feet. Use the Law of Cosines to determine each of the three angles.

Appendix – Proof of the Law of Sines

We will use what we know about right triangles to prove the Law of Sines. The key idea is to create right triangles from the diagram for a general triangle by drawing an altitude of length h from one of the vertices. We first note that if α, β, and γ are the three angles of a triangle, then

$$\alpha + \beta + \gamma = 180°.$$

This means that at most one of the three angles can be an obtuse angle (between $90°$ and $180°$), and hence, at least two of the angles must be acute (less than $90°$). Figure 3.17 shows the two possible cases for a general triangle. The triangle on the left has three acute angles and the triangle on the right has two acute angles (α and β) and one obtuse angle (γ).

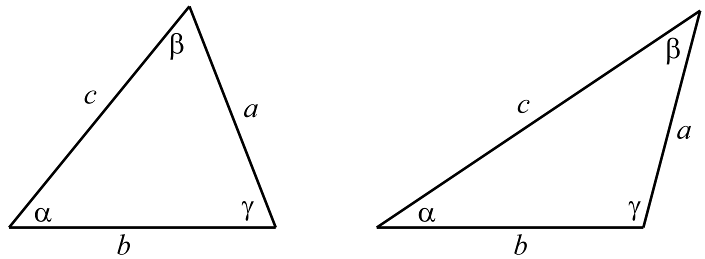


Figure 3.17: General Triangles

We will now prove the Law of Sines for the case where all three angles of the triangle are acute angles. The proof for the case where one angle of the triangle is obtuse is included in the exercises. The key idea is to create right triangles from the diagram for a general triangle by drawing altitudes in the triangle as shown in Figure 3.18 where an altitude of length h is drawn from the vertex of angle β and an altitude of length k is drawn from the vertex of angle γ.

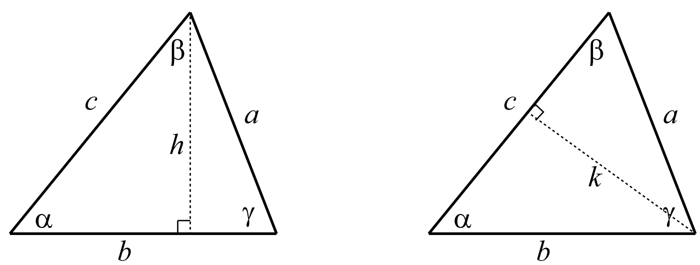


Figure 3.18: Diagram for the Proof of the Law of Sines

Using the right triangles in the diagram on the left, we see that

$$\sin(\alpha) = \frac{h}{c} \qquad\qquad \sin(\gamma) = \frac{h}{a}$$

From this, we can conclude that

$$h = c\sin(\alpha) \qquad\qquad h = a\sin(\gamma) \tag{1}$$

Using the two equations in (1), we can use the fact that both of the right sides are equal to h to conclude that

$$c\sin(\alpha) = a\sin(\gamma).$$

Now, dividing both sides of the last equation by ac, we see that

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}. \tag{2}$$

We now use a similar argument using the triangle on the right in Figure 3.18. We see that

$$\sin(\alpha) = \frac{k}{b} \qquad \sin(\beta) = \frac{k}{a}$$

From this, we obtain

$$k = b\sin(\alpha) \qquad k = a\sin(\beta)$$

and so

$$\begin{aligned} b\sin(\alpha) &= a\sin(\beta) \\ \frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} \end{aligned} \tag{3}$$

We can now use equations (2) and (3) to complete the proof of the Law of Sines, which is

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

Appendix – Proof of the Law of Cosines

As with the Law of Sines, we will use results about right triangles to prove the Law of Cosines. We will also use the distance formula. We will start with a general triangle with a, b, and c representing the lengths of the sides opposite the angles α, β, and γ, respectively. We will place the angle γ in standard position in the coordinate system as shown in Figure 3.19.

In this diagram, the angle γ is shown as an obtuse angle but the proof would be the same if γ was an acute angle. We have labeled the vertex of angle α as A with coordinates (x, y) and we have drawn a line from A perpendicular to the x-axis. So from the definitions of the trigonometric functions in Section 3.1, we see that

$$\begin{aligned} \cos(\gamma) &= \frac{x}{b} & \sin(\gamma) &= \frac{y}{b} \\ x &= b\cos(\gamma) & y &= b\sin(\gamma) \end{aligned} \tag{4}$$

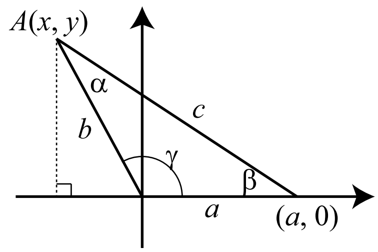


Figure 3.19: Diagram for the Law of Cosines

We now use the distance formula with the points A and the vertex of angle β, which has coordinates $(a, 0)$. This gives

$$\begin{aligned} c &= \sqrt{(x-a)^2 + (y-0)^2} \\ c^2 &= (x-a)^2 + y^2 \\ c^2 &= x^2 - 2ax + a^2 + y^2 \end{aligned}$$

We now substitute the values for x and y in equation (4) and obtain

$$\begin{aligned} c^2 &= b^2\cos^2(\gamma) - 2ab\cos(\gamma) + a^2 + b^2\sin^2(\gamma) \\ c^2 &= a^2 + b^2\cos^2(\gamma) + b^2\sin^2(\gamma) - 2ab\cos(\gamma) \\ c^2 &= a^2 + b^2\left(\cos^2(\gamma) + \sin^2(\gamma)\right) - 2ab\cos(\gamma) \end{aligned}$$

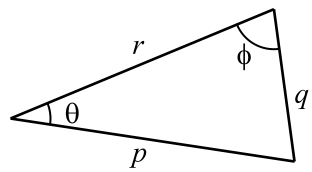
We can now use the last equation and the fact that $\cos^2(\gamma) + \sin^2(\gamma) = 1$ to conclude that

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

This proves one of the equations in the Law of Cosines. The other two equations can be proved in the same manner by placing each of the other two angles in standard position.

Summary of Section 3.3

In this section, we studied the following important concepts and ideas:
The Law of Sines and the Law of Cosines can be used to determine the lengths of sides of a triangle and the measure of the angles of a triangle.



The **Law of Sines** states that if q is the length of the side opposite the angle θ and p is the length of the side opposite the angle ϕ, then

$$\frac{\sin(\theta)}{q} = \frac{\sin(\phi)}{p}.$$

The **Law of Cosines** states that if p, q, and r are the lengths of the sides of a triangle and θ is the angles opposite the side q, then

$$q^2 = p^2 + r^2 - 2pr\cos(\theta).$$

Each of the equations in the Law of Sines and the Law of Cosines involves four variables. So if we know the values of three of the variables, then we can use the appropriate equation to solve for the fourth variable.

Exercises for Section 3.3

For Exercises (1) through (4), use the Law of Sines.

⋆ **1.** Two angles of a triangle are 42° and 73°. The side opposite the 73° angle is 6.5 feet long. Determine the third angle of the triangle and the lengths of the other two sides.

2. A triangle has a side that is 4.5 meters long and this side is adjacent to an angle of 110°. In addition, the side opposite the 110° angle is 8 meters long. Determine the other two angles of the triangle and the length of the third side.

⋆ **3.** A triangle has a side that is 5 inches long that is adjacent to an angle of 61°. The side opposite the 61° angle is 4.5 inches long. Determine the other two angles of the triangle and the length of the third side.

4. In a given triangle, the side opposite an angle of 107° is 18 inches long. One of the sides adjacent to the 107° angle is 15.5 inches long. Determine the other two angles of the triangle and the length of the third side.

For Exercises (5) through (6), use the Law of Cosines.

⋆ **5.** The three sides of a triangle are 9 feet long, 5 feet long, and 7 feet long. Determine the three angles of the triangle.

6. A triangle has two sides of lengths 8.5 meters and 6.8 meters. The angle formed by these two sides is 102°. Determine the length of the third side and the other two angles of the triangle.

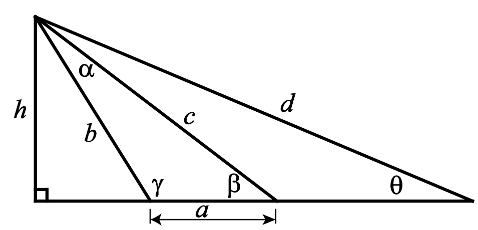
For the remaining exercises, use an appropriate method to solve the problem.

7. Two angles of a triangle are 81.5° and 34°. The length of the side opposite the third angle is 8.8 feet. Determine the third angle and the lengths of the other two sides of the triangle.

8. In the diagram to the right, determine the value of γ (to the nearest hundredth of a degree) and determine the values of h and d (to the nearest thousandth) if it is given that

$a = 4 \qquad b = 8$

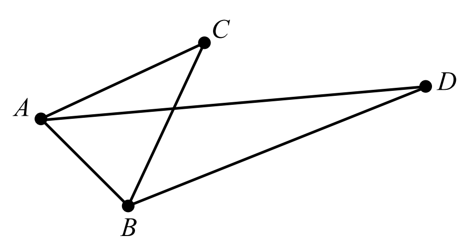
$c = 10 \qquad \theta = 26°$



9. In the diagram to the right, it is given that:

- The length of AC is 2.
- The length of BC is 2.
- $\angle ACB = 40°$.
- $\angle CAD = 20°$.
- $\angle CBD = 45°$.

Determine the lengths of AB and AD to the nearest thousandth.



3.4 Applications of Triangle Trigonometry

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we use the Law of Sines and the Law of Cosines to help solve applied problems that involve triangles?
- How do we determine the area of a triangle?
- What is Heron's Law for the area of a triangle?

In Section 3.2, we used right triangles to solve some applied problems. It should then be no surprise that we can use the Law of Sines and the Law of Cosines to solve applied problems involving triangles that are not right triangles.

In most problems, we will first draw a rough diagram or picture showing the triangle or triangles involved in the problem. We then need to label the known quantities. Once that is done, we can see if there is enough information to use the Law of Sines or the Law of Cosines. Remember that each of these laws involves four quantities. If we know the value of three of those four quantities, we can use that law to determine the fourth quantity.

We begin with the example in Progress Check 3.12. The solution of this problem involved some complicated work with right triangles and some algebra. We will now solve this problem using the results from Section 3.3.

Example 3.19 (Height to the Top of a Flagpole)
Suppose that the flagpole sits on top a hill and that we cannot directly measure the length of the shadow of the flagpole as shown in Figure 3.19.

Some quantities have been labeled in the diagram. Angles α and β are angles of elevation to the top of the flagpole from two different points on level ground. These points are d feet apart and directly in line with the flagpole. The problem is to determine h, the height from level ground to the top of the flagpole. The following measurements have been recorded.

$$\alpha = 43.2^\circ \qquad d = 22.75\text{feet}$$
$$\beta = 34.7^\circ$$

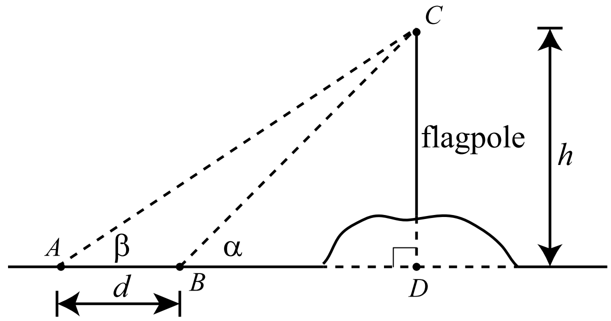


Figure 3.20: Flagpole on a hill

We notice that if we knew either length BC or BD in $\triangle BDC$, then we could use right triangle trigonometry to determine the length CD, which is equal to h. Now look at $\triangle ABC$. We are given the measure of angle β. However, we also know the measure of angle α. Because they form a straight angle, we have

$$\angle ABC + \alpha = 180°.$$

Hence, $\angle ABC = 180° - 43.2° = 136.8°$. We now know two angles in $\triangle ABC$ and hence, we can determine the third angle as follows:

$$\begin{aligned} \beta + \angle ABC + \angle ACB &= 180° \\ 34.7° + 136.8° + \angle ACB &= 180° \\ \angle ACB &= 8.5° \end{aligned}$$

We now know all angles in $\triangle ABC$ and the length of one side. We can use the Law of Sines. We have

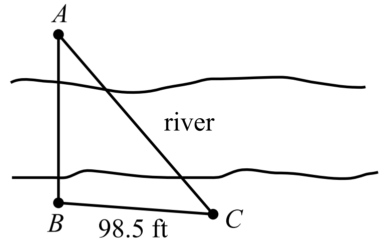
$$\begin{aligned} \frac{BC}{\sin(34.7°)} &= \frac{22.75}{\sin(8.5°)} \\ BC &= \frac{22.75\sin(34.7°)}{\sin(8.5°)} \approx 87.620 \end{aligned}$$

We can now use the right triangle $\triangle BDC$ to determine h as follows:

$$\begin{aligned} \frac{h}{BC} &= \sin\left(43.2°\right) \\ h &= BC \cdot \sin\left(43.2°\right) \approx 59.980 \end{aligned}$$

So the top of the flagpole is 59.980 feet above the ground. This is the same answer we obtained in Progress Check 3.12.

Progress Check 3.20 (An Application)
A bridge is to be built across a river. The bridge will go from point A to point B in the diagram on the right. Using a transit (an instrument to measure angles), a surveyor measures angle ABC to be $94.2°$ and measures angle BCA to be $48.5°$. In addition, the distance from B to C is measured to be 98.5 feet. How long will the bridge from point B to point A be?



Area of a Triangle

We will now develop a few different ways to calculate the area of a triangle. Perhaps the most familiar formula for the area is the following:

> The area A of a triangle is
> $$A = \frac{1}{2}bh,$$
> where b is the length of the base of a triangle and h is the length of the altitude that is perpendicular to that base.

The triangles in Figure 3.21 illustrate the use of the variables in this formula.

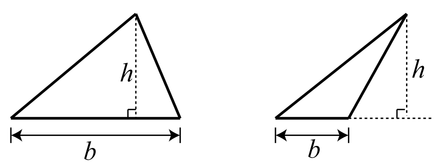
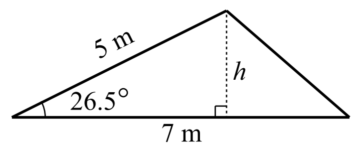


Figure 3.21: Diagrams for the Formula for the Area of a Triangle

A proof of this formula for the area of a triangle depends on the formula for the area of a parallelogram and is included in Appendix C.

Progress Check 3.21 (The Area of a Triangle)

Suppose that the length of two sides of a triangle are 5 meters and 7 meters and that the angle formed by these two sides is 26.5°. See the diagram on the right.



For this problem, we are using the side of length 7 meters as the base. The altitude of length h that is perpendicular to this side is shown.

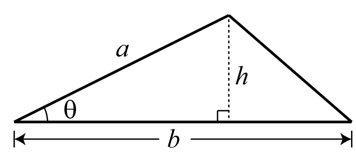
1. Use right triangle trigonometry to determine the value of h.
2. Determine the area of this triangle.

One purpose of Progress Check 3.21 was to illustrate that if we know the length of two sides of a triangle and the angle formed by these two sides, then we can determine the area of that triangle.

The Area of a Triangle
The area of a triangle equals one-half the product of two of its sides times the sine of the angle formed by these two sides.

Progress Check 3.22 (Proof of the Formula for the Area of a Triangle)

In the diagram on the right, b is the length of the base of a triangle, a is the length of another side, and θ is the angle formed by these two sides. We let A be the area of the triangle.



Follow the procedure illustrated in Progress Check 3.21 to prove that

$$A = \frac{1}{2}ab\sin(\theta).$$

Explain why this proves the formula for the area of a triangle.

There is another common formula for the area of a triangle known as Heron's Formula named after Heron of Alexandria (circa 75 CE). This formula shows that the area of a triangle can be computed if the lengths of the three sides of the triangle are known.

Heron's Formula
The area A of a triangle with sides of length a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

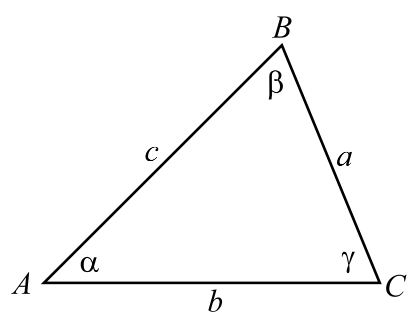
For example, suppose that the lengths of the three sides of a triangle are $a = 3$ ft, $b = 5$ ft, and $c = 6$ ft. Using Heron's Formula, we get

$$s = \frac{1}{2}(a+b+c) \qquad A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$s = 7 \qquad A = \sqrt{7(7-3)(7-5)(7-6)}$$
$$A = \sqrt{42}$$

This fairly complex formula is actually derived from the previous formula for the area of a triangle and the Law of Cosines. We begin our exploration of the proof of this formula in Progress Check 3.23.

Progress Check 3.23 (Heron's Formula)

Suppose we have a triangle as shown in the diagram on the right.



1. Use the Law of Cosines that involves the angle γ and solve this formula for $\cos(\gamma)$. This gives a formula for $\cos(\gamma)$ in terms of a, b, and c.

2. Use the Pythagorean Identity $\cos^2(\gamma) + \sin^2(\gamma) = 1$ to write $\sin(\gamma)$ in terms of $\cos^2(\gamma)$. Substitute for $\cos^2(\gamma)$ using the formula in part (1). This gives a formula for $\sin(\gamma)$ in terms of a, b, and c. (Do not do any algebraic implication.)

3. We also know that a formula for the area of this triangle is $A = \frac{1}{2}ab\sin(\gamma)$. Substitute for $\sin(\gamma)$ using the formula in (2). (Do not do any algebraic simplification.) This gives a formula for the area A in terms of a, b, and c.

The formula obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}$$

This is a formula for the area of a triangle in terms of the lengths of the three sides of the triangle. It does not look like Heron's Formula, but we can use some substantial algebra to rewrite this formula to obtain Heron's Formula. This algebraic work is completed in the appendix for this section.

Appendix – Proof of Heron's Formula

The formula for the area of a triangle obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}$$

We now complete the algebra to show that this is equivalent to Heron's formula. The first step is to rewrite the part under the square root sign as a single fraction.

$$\begin{aligned} A &= \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} \\ &= \frac{1}{2}ab\sqrt{\frac{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}{(2ab)^2}} \\ &= \frac{1}{2}ab\frac{\sqrt{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}}{2ab} \\ &= \frac{\sqrt{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}}{4} \end{aligned}$$

Squaring both sides of the last equation, we obtain

$$A^2 = \frac{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}{16}.$$

The numerator on the right side of the last equation is a difference of squares. We will now use the difference of squares formula, $x^2 - y^2 = (x - y)(x + y)$ to factor the numerator.

$$\begin{aligned} A^2 &= \frac{(2ab)^2 - \left(a^2+b^2-c^2\right)^2}{16} \\ &= \frac{\left(2ab - \left(a^2+b^2-c^2\right)\right)\left(2ab + \left(a^2+b^2-c^2\right)\right)}{16} \\ &= \frac{\left(-a^2+2ab-b^2+c^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} \end{aligned}$$

We now notice that $-a^2+2ab-b^2 = -(a-b)^2$ and $a^2+2ab+b^2 = (a+b)^2$. So using these in the last equation, we have

$$\begin{aligned} A^2 &= \frac{\left(-(a-b)^2+c^2\right)\left((a+b)^2-c^2\right)}{16} \\ &= \frac{\left(-\left[(a-b)^2-c^2\right]\right)\left((a+b)^2-c^2\right)}{16} \end{aligned}$$

We can once again use the difference of squares formula as follows:

$$\begin{aligned} (a-b)^2-c^2 &= (a-b-c)(a-b+c) \\ (a+b)^2-c^2 &= (a+b-c)(a+b+c) \end{aligned}$$

Substituting this information into the last equation for A^2, we obtain

$$A^2 = \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16}.$$

Since $s = \frac{1}{2}(a+b+c)$, $2s = a+b+c$. Now notice that

$$\begin{aligned} -(a-b-c) &= -a+b+c \\ &= a+b+c-2a \\ &= 2s-2a \end{aligned} \qquad \begin{aligned} a-b+c &= a+b+c-2b \\ &= 2s-2b \end{aligned}$$

$$\begin{aligned} a+b-c &= a+b+c-2c \\ &= 2s-2c \end{aligned} \qquad a+b+c = 2s$$

So

$$\begin{aligned}
A^2 &= \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \\
&= \frac{(2s-2a)(2s-2b)(2s-2c)(2s)}{16} \\
&= \frac{16s(s-a)(s-b)(s-c)}{16} \\
&= s(s-a)(s-b)(s-c) \\
A &= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

This completes the proof of Heron's formula.

Summary of Section 3.4

In this section, we studied the following important concepts and ideas:

- How to use right triangle trigonometry, the Law of Sines, and the Law of Cosines to solve applied problems involving triangles.

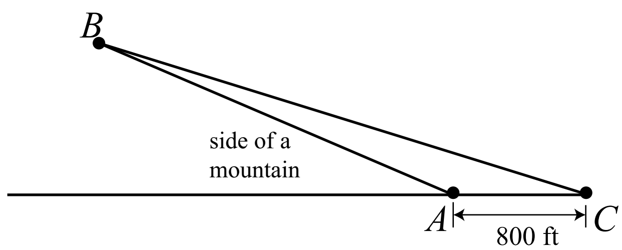
- Three ways to determine the area A of a triangle.

  * $A = \frac{1}{2}bh$, where b is the length of the base and h is the length of the altitude.
  * $A = \frac{1}{2}ab\sin(\theta)$, where a and b are the lengths of two sides of the triangle and θ is the angle formed by the sides of length a and b.
  * **Heron's Formula**. If a, b, and c are the lengths of the sides of a triangle and $s = \frac{1}{2}(a+b+c)$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

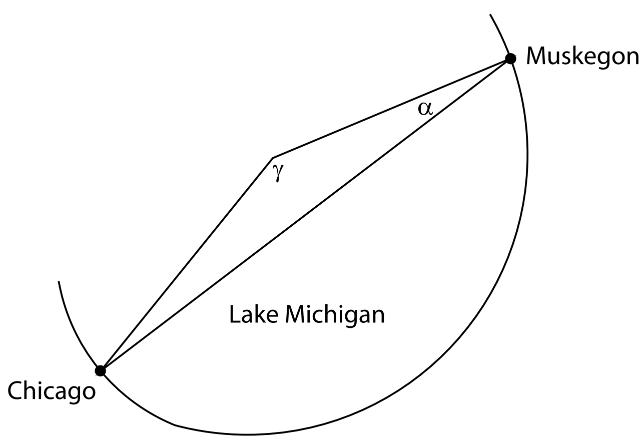
Exercises for Section 3.4

⋆ **1.** A ski lift is to be built along the side of a mountain from point A to point B in the following diagram. We wish to determine the length of this ski lift.



A surveyor determines the measurement of angle BAC to be $155.6°$ and then measures a distance of 800 feet from Point A to Point C. Finally, she determines the measurement of angle BCA to be $17.2°$. What is the length of the ski lift (from point A to point B)?

⋆ **2.** A boat sails from Muskegon bound for Chicago, a sailing distance of 121 miles. The boat maintains a constant speed of 15 miles per hour. After encountering high cross winds the crew finds itself off course by $20°$ after 4 hours. A crude picture is shown in the following diagram, where $\alpha = 20°$.

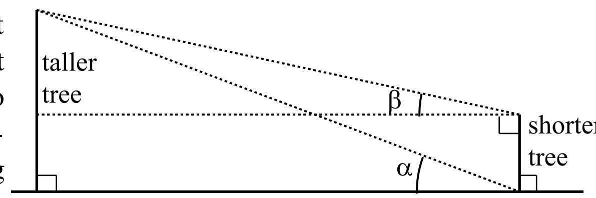


(**a**) How far is the sailboat from Chicago at this time?

(**b**) What is the degree measure of the angle γ (to the nearest tenth) in the diagram? Through what angle should the boat turn to correct its course and be heading straight to Chicago?

(**c**) Assuming the boat maintains a speed of 15 miles per hour, how much time have they added to their trip by being off course?

3. Two trees are on opposite sides of a river. It is known that the height of the shorter of the two trees is 13 meters. A person makes the following angle measurements:

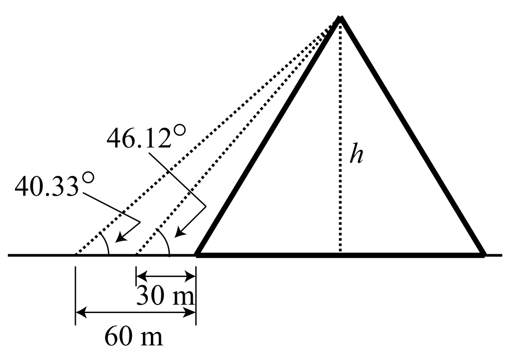


- The angle of elevation from the base of the shorter tree to the top of the taller tree is $\alpha = 20°$.
- The angle of elevation from the top of the shorter tree to the top of the taller tree is $\beta = 12°$.

Determine the distance between the bases of the two trees and the height of the taller tree.

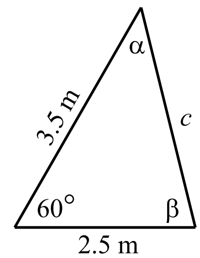
4. One of the original Seven Wonders of the World, the Great Pyramid of Giza (also known as the Pyramid of Khufu or the Pyramid of Cheops), was believed to have been built in a 10 to 20 year period concluding around 2560 B.C.E. It is also believed that the original height of the pyramid was 146.5 meters but that it is now shorter due to erosion and the loss of some topmost stones. [1]

To determine its current height, the angle of elevation from a distance of 30 meters from the base of the pyramid was measured to be 46.12°, and then the angle of elevation was measured to be 40.33° from a distance of 60 meters from the base of the pyramid as shown in the following diagram. Use this information to determine the height h of the pyramid. (138.8 meters)

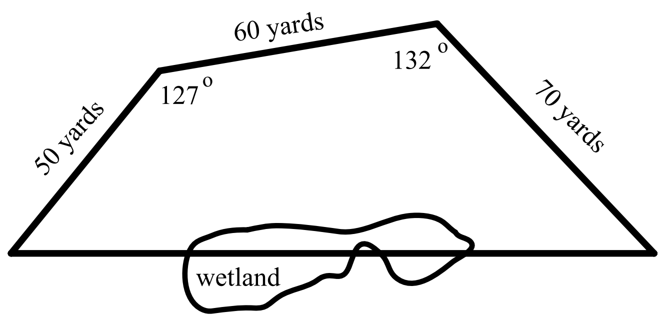


[1] https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza

5. Two sides of a triangle have length 2.5 meters and 3.5 meters, and the angle formed by these two sides has a measure of 60°. Determine the area of the triangle. **Note**: This is the triangle in Progress Check 3.17 on page 200.



6. A field has the shape of a quadrilateral that is not a parallelogram. As shown in the following diagram, three sides measure 50 yards, 60 yards, and 70 yards. Due to some wetland along the fourth side, the length of the fourth side could not be measured directly. Two angles shown in the diagram measure 127° and 132°.



Determine the length of the fourth side of the quadrilateral, the measures of the other two angles in the quadrilateral, and the area of the quadrilateral. Lengths must be accurate to the nearest hundredth of a yard, angle measures must be correct to the nearest hundredth of a degree, and the area must be correct to the nearest hundredth of a square yard.

3.5 Vectors from a Geometric Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a vector?
- How do we use the geometric form of vectors to find the sum of two vectors?
- How do we use the geometric form of vectors to find a scalar multiple of a vector?
- How do we use the geometric form of vectors to find the difference of two vectors?
- What is the angle between two vectors?
- Why is force a vector and how do we use vectors and triangles to determine forces acting on an object?

We have all had the experience of dropping something and watching it fall to the ground. What is happening, of course, is that the force of gravity is causing the object to fall to the ground. In fact, we experience the force of gravity everyday simply by being on Earth. Each person's weight is a measure of the force of gravity since pounds are a unit of force. So when a person weighs 150 pounds, it means that gravity is exerting a force of 150 pounds straight down on that person. Notice that we described this with a quantity and a direction (straight down). Such a quantity (with magnitude and direction) is called a vector.

Now suppose that person who weighs 150 pounds is standing on a hill. In mathematics, we simplify the situation and say that the person is standing on an inclined plane as shown in Figure 3.22. (By making the hill a straight line, we simplify the mathematics involved.) In the diagram in Figure 3.22, an object is on the inclined plane at the point P. The inclined plane makes an angle of θ with the horizontal. The vector $\mathbf{w}$ shows the weight of the object (force of gravity, straight down). The diagram also shows two other vectors. The vector $\mathbf{b}$ is perpendicular

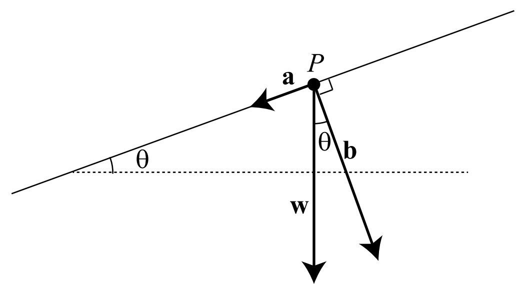


Figure 3.22: Inclined Plane

to the plane and represents the force that the object exerts on the plane. The vector **a** is perpendicular to **b** and parallel to the inclined plane. This vector represents the force of gravity along the plane. In this and the next section, we will learn more about these vectors and how to determine the magnitudes of these vectors. We will also see that with our definition of the addition of two vectors that $\mathbf{w} = \mathbf{a} + \mathbf{b}$.

Definitions

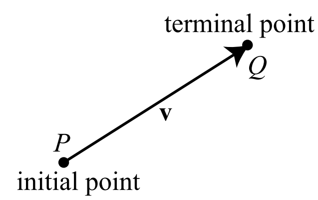
There are some quantities that require only a number to describe them. We call this number the magnitude of the quantity. One such example is temperature since we describe this with only a number such as 68 degrees Fahrenheit. Other such quantities are length, area, and mass. These types of quantities are often called **scalar quantities**. However, there are other quantities that require both a magnitude and a direction. One such example is force, and another is velocity. We would describe a velocity with something like 45 miles per hour northwest. Velocity and force are examples of a **vector quantity**. Other examples of vectors are acceleration and displacement.

Some vectors are closely associated with scalars. In mathematics and science, we make a distinction between *speed* and *velocity*. Speed is a scalar and we would say something like our speed is 65 miles per hour. However, if we used a velocity, we would say something like 65 miles per hour east. This is different than a velocity of 65 miles per hour north even though in both cases, the speed is 65 miles per hour.

Definition. A **vector** is a quantity that has both magnitude and direction. A **scalar** is a quantity that has magnitude only.

Geometric Representation of Vectors

Vectors can be represented geometrically by arrows (directed line segments). The **arrowhead** indicates the direction of the vector, and the **length** of the arrow describes the magnitude of the vector. A vector with initial point P (the tail of the arrow) and terminal point Q (the tip of the arrowhead) can be represented by



$$\overrightarrow{PQ}, \mathbf{v}, \text{ or } \overrightarrow{v}.$$

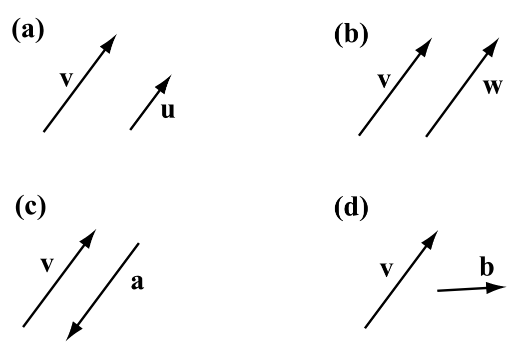
We often write $\mathbf{v} = \overrightarrow{PQ}$. In this text, we will use boldface font to designate a vector. When writing with pencil and paper, we always use an arrow above the letter (such as $\overrightarrow{v}$) to designate a vector. The **magnitude** (or **norm** or **length**) of the vector $\mathbf{v}$ is designated by $|\mathbf{v}|$. It is important to remember that $|\mathbf{v}|$ is a number that represents the magnitude or length of the vector $\mathbf{v}$.

According to our definition, a vector possesses the attributes of length (magnitude) and direction, but position is not mentioned. So we will consider two vectors to be equal if they have the same magnitude and direction. For example, if two different cars are both traveling at 45 miles per hour northwest (but in different locations), they have equal velocity vectors. We make a more formal definition.

Definition. Two vectors are **equal** if and only if they have the same magnitude and the same direction. When the vectors $\mathbf{v}$ and $\mathbf{w}$ are equal, we write $\mathbf{v} = \mathbf{w}$.

Progress Check 3.24 (Equal and Unequal Vectors)

In each of the following diagrams, a vector $\mathbf{v}$ is shown next to four other vectors. Which (if any) of these four vectors are equal to the vector $\mathbf{v}$?



Operations on Vectors

Scalar Multiple of a Vector

Doubling a scalar quantity is simply a matter of multiplying its magnitude by 2. For example, if a container has 20 ounces of water and the amount of water is doubled, it will then have 40 ounces of water. What do we mean by doubling a vector? The basic idea is to keep the same direction and multiply the magnitude by 2. So if an object has a velocity of 5 feet per second southeast and a second object has a velocity of twice that, the second object will have a velocity of 10 feet per second in the southeast direction. In this case, we say that we multiplied the vector by the scalar 2. We now make a definition that also takes into account that a scalar can be negative.

Definition. For any vector $\mathbf{v}$ and any scalar c, the vector $c\mathbf{v}$ (called a **scalar multiple** of the vector $\mathbf{v}$) is a vector whose magnitude is $|c|$ times the magnitude of the vector $\mathbf{v}$. That is,

$$|c\mathbf{v}| = |c||\mathbf{v}|.$$

Note: In this equation, $|c|$ is the absolute value of the scalar c. Care must be taken not to confuse this with the notation $|\mathbf{v}|$, which is the magnitude of the vector $\mathbf{v}$. This is one reason it is important to have a notation that clearly indicates when we are working with a vector or a scalar. Also, using this definition, we see that

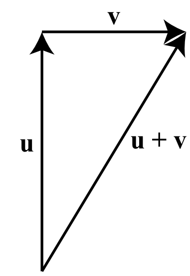
- If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
- If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
- If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.

The vector $\mathbf{0}$ is called the **zero vector** and the zero vector has no magnitude and no direction. We sometimes write $\overrightarrow{0}$ for the zero vector.

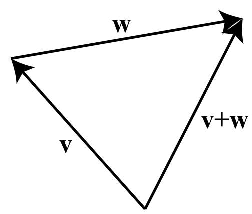
Addition of Vectors

We illustrate how to add vectors with two displacement vectors. As with velocity and speed, there is a distinction between displacement and distance. **Distance** is a scalar. So we might say that we have traveled 2 miles. **Displacement**, on the other hand, is a vector consisting of a distance and a direction. So the vectors 2 miles north and 2 miles east are different displacement vectors.

Now if we travel 3 miles north and then travel 2 miles east, we end at a point that defines a new displacement vector. See the diagram to the right. In this diagram, **u** is "3 miles north" and **v** is "2 miles east." The vector sum **u** + **v** goes from the initial point of **u** to the terminal point of **v**.



Definition. The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of **w** coincides with the terminal point of **v**. The vector **v** + **w** is the vector whose initial point coincides with the initial point of **v** and whose terminal point coincides with the terminal point of **w**.



The vector **v** + **w** is called the **sum** or **resultant** of the vectors **v** and **w**.

In the definition, notice that the vectors **v**, **w**, and **v** + **w** are placed so that the result is a triangle. The lengths of the sides of that triangle are the magnitudes of these sides |**v**|, |**w**|, and |**v** + **w**|. If we place the two vectors **v** and **w** so that their initial points coincide, we can use a parallelogram to add the two vectors. This is shown in Figure 3.23.

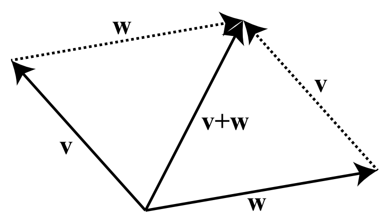
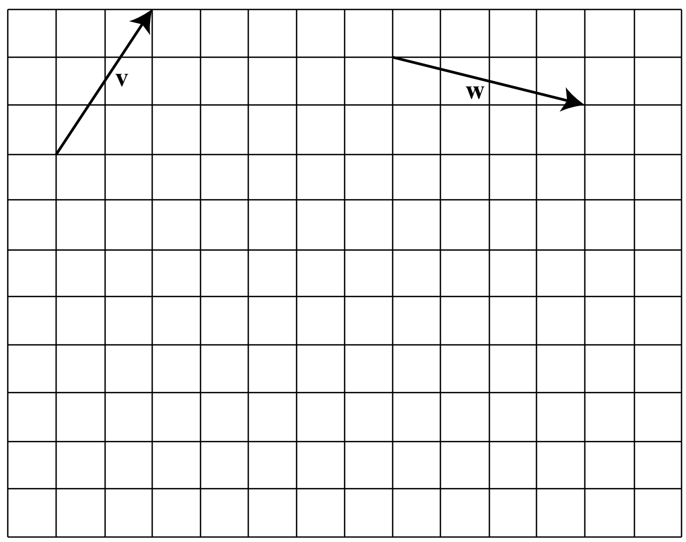


Figure 3.23: Sum of Two Vectors Using a Parallelogram

Notice that the vector **v** forms a pair of opposite sides of the parallelogram as does the vector **w**.

Progress Check 3.25 (Operations on Vectors)
The following diagram shows two vectors, $\mathbf{v}$ and $\mathbf{w}$. Draw the following vectors:

(a) $\mathbf{v}+\mathbf{w}$ **(b)** $2\mathbf{v}$ **(c)** $2\mathbf{v}+\mathbf{w}$ **(d)** $-2\mathbf{w}$ **(e)** $-2\mathbf{w}+\mathbf{v}$



Subtraction of Vectors

Before explaining how to subtract vectors, we will first explain what is meant by the "negative of a vector." This works similarly to the negative of a real number. For example, we know that when we add -3 to 3, the result is 0. That is, $3+(-3)=0$.

We want something similar for vectors. For a vector $\mathbf{w}$, the idea is to use the scalar multiple $(-1)\mathbf{w}$. The vector $(-1)\mathbf{w}$ has the same magnitude as $\mathbf{w}$ but has the opposite direction of $\mathbf{w}$. We define $-\mathbf{w}$ to be $(-1)\mathbf{w}$. Figure shows that when we add $-\mathbf{w}$ to $\mathbf{w}$, the terminal point of the sum is the same as the initial point of the sum and so the result is the zero vector. That is, $\mathbf{w}+(-\mathbf{w})=\mathbf{0}$.

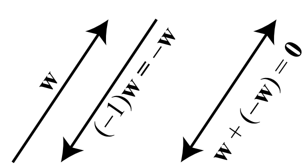
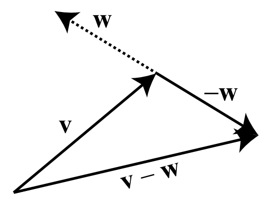


Figure 3.24: The Sum of a Vector and Its Negative

We are now in a position to define subtraction of vectors. The idea is much the same as subtraction of real numbers in that for any two real numbers a and b, $a - b = a + (-b)$.

Definition. For any two vectors **v** and **w**, the **difference between v and w** is denoted by **v** − **w** and is defined as follows:

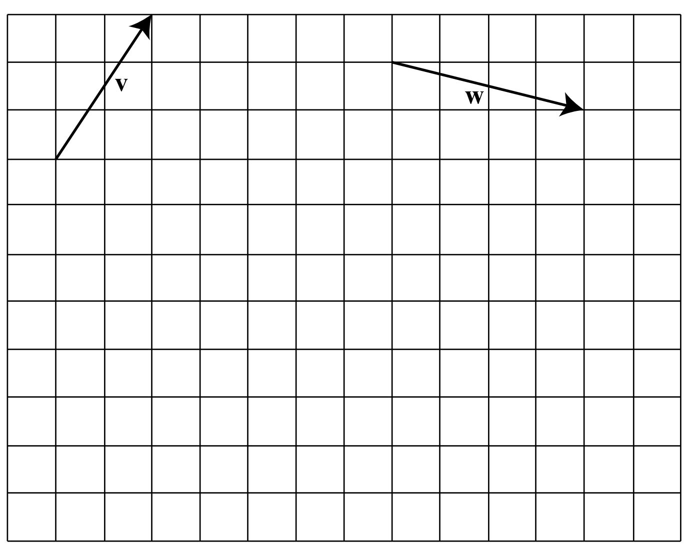
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$



We also say that we are subtracting the vector **w** from the vector **v**.

Progress Check 3.26 (Operations on Vectors)
The following diagram shows two vectors, **v** and **w**. Draw the following vectors:

(a) −**w** **(b) v** − **w** **(c)** −**v** **(d) w** − **v**



The Angle Between Two Vectors

We have seen that we can use triangles to help us add or subtract two vectors. The lengths of the sides of the triangle are the magnitudes of certain vectors. Since we are dealing with triangles, we will also use angles determined by the vectors.

Definition. The angle θ between vectors is the angle formed by these two vectors (with $0° \le \theta \le 180°$) when they have the same initial point.

So in the diagram on the left in Figure 3.25, the angle θ is the angle between the vectors **v** and **w**. However, when we want to determine the sum of two angles, we often form the parallelogram determined by the two vectors as shown in the diagram on the right in Figure 3.25. (See page 427 in Appendix C for a summary of properties of a parallelogram.) We then will use the angle $180° - \theta$ and the Law of Cosines since we the two sides of the triangle are the lengths of **v** and **w** and the included angle is $180° - \theta$. We will explore this in the next progress check.

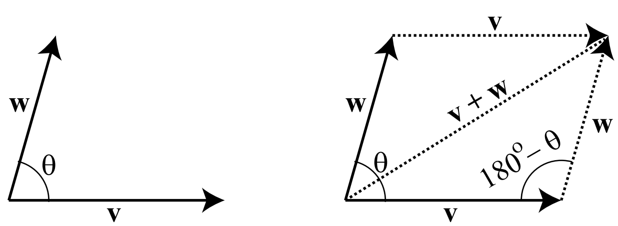


Figure 3.25: Angle Between Two Vectors

Progress Check 3.27 (The Sum of Two Vectors)
Suppose that the vectors **a** and **b** have magnitudes of 80 and 60, respectively, and that the angle θ between the two vectors is 53 degrees. In Figure 3.26, we have drawn the parallelogram determined by these two vectors and have labeled the vertices for reference.

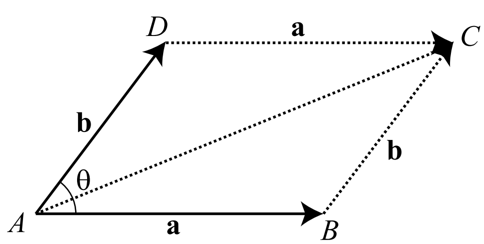


Figure 3.26: Diagram for Progress Check 3.27

Remember that a vector is determined by its magnitude and direction. We will determine $|\mathbf{a} + \mathbf{b}|$ and the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$.

1. Determine the measure of $\angle ABC$.

2. In $\triangle ABC$, the length of side AB is $|\mathbf{a}| = 80$ and the length of side BC is $|\mathbf{b}| = 60$. Use this triangle and the Law of Cosines to determine the length of the third side, which is $|\mathbf{a} + \mathbf{b}|$.

3. Determine the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$. This is $\angle CAB$ in $\triangle ABC$.

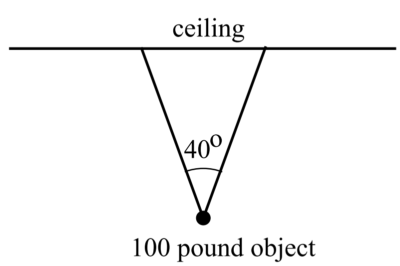
Force

An important vector quantity is that of force. In physics, a force on an object is defined as any interaction that, when left un-opposed, will change the motion of the object. So a force will cause an object to change its velocity, that is the object will accelerate. More informally, a force is a push or a pull on an object.

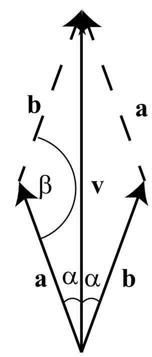
One force that affects our lives is the force of gravity. The magnitude of the force of gravity on a person is that person's weight. The direction of the force of gravity is straight down. So if a person who weighs 150 pounds is standing still on the ground, then the ground is also exerting a force of 150 pounds on the person in the upward direction. The net force on the stationary person is zero. This is an example of what is known as *static equilibrium*. When an object is in static equilibrium, the sum of the forces acting on the object is equal to the zero vector.

Example 3.28 (Object Suspended from a Ceiling)

Suppose a 100 pound object is suspended from the ceiling by two wires that form a 40° angle as shown in the diagram to the right. Because the object is stationary, the two wires must exert a force on the object so that the sum of these two forces is equal to 100 pounds straight up. (The force of gravity is 100 pounds straight down.)



We will assume that the two wires exert forces of equal magnitudes and that the angle between these forces and the vertical in 20°. So our first step is to draw a picture of these forces, which is shown on the right. The vector **v** is a vector of magnitude 100 pounds. The vectors **a** and **b** are the vectors for the forces exerted by the two wires. (We have $|\mathbf{a}| = |\mathbf{b}|$.) We also know that $\alpha = 20°$ and so $2\alpha = 40°$. So the angle between the vectors **a** and **b** is 40° and so by the properties of parallelograms, $\beta = 140°$. (See page 427.)



Progress Check 3.29 (Completion of Example 3.28)
Use triangle trigonometry to determine the magnitude of the vector **a** in Example 3.28. Note that we already know the direction of this vector.

Inclined Planes

At the beginning of this section, we discussed the forces involved when an object is place on an inclined plane. Figure 3.27 is the diagram we used, but we now have added labels for some of the angles. Recall that the vector **w** shows the weight of the object (force of gravity, straight down), the vector **b** is perpendicular to the plane and represents the force that the object exerts on the plane, and the vector **a** is perpendicular to **b** and parallel to the inclined plane. This vector represents the force of gravity along the plane. Notice that we have also added a second copy of the vector **a** that begins at the tip of the vector **b**.

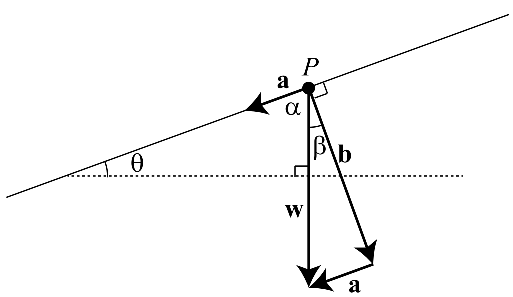


Figure 3.27: Inclined Plane

Using the angles shown, we see that $\alpha + \beta = 90°$ since they combine to form a right angle, and $\alpha + \theta = 90°$ since they are the two acute angles in a right triangle. From this, we conclude that $\beta = \theta$. This gives us the final version of the diagram of the forces on an inclined plane shown in Figure 3.28. Notice that the vectors **a**,

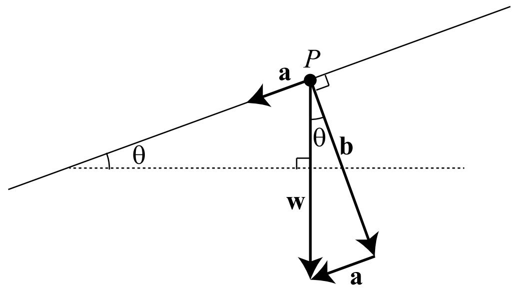


Figure 3.28: Inclined Plane

b, and **w** form a right triangle, and so we can use right triangle trigonometry for problems dealing with the forces on an inclined plane.

Progress Check 3.30 (A Problem Involving an Inclined Plane)
An object that weighs 250 pounds is placed on an inclined plane that makes an angle of 12° degrees with the horizontal. Using a diagram like the one in Figure 3.28, determine the magnitude of the force against the plane caused by the object and the magnitude of the force down the plane on the object due to gravity. **Note**: The magnitude of the force down the plane will be the force in the direction up the plane that is required to keep the object stationary.

Summary of Section 3.5

In this section, we studied the following important concepts and ideas:

Vectors and Scalars

A **vector** is a quantity that has both magnitude and direction. A **scalar** is a quantity that has magnitude only. Two vectors are **equal** if and only if they have the same magnitude and the same direction.

Scalar Multiple of a Vector

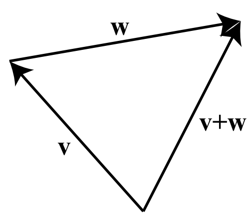
For any vector **v** and any scalar c, the vector $c\mathbf{v}$ (called a **scalar multiple** of the vector **v**) is a vector whose magnitude is $|c|$ times the magnitude of the vector **v**.

- If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
- If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
- If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.
- Using vector notation, we have $|c\mathbf{v}| = |c||\mathbf{v}|$.

The vector $\mathbf{0}$ is called the **zero vector** and the zero vector has no magnitude and no direction. We sometimes write $\overrightarrow{0}$ for the zero vector.

The Sum of Two Vectors

The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of $\mathbf{w}$ coincides with the terminal point of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is the vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.



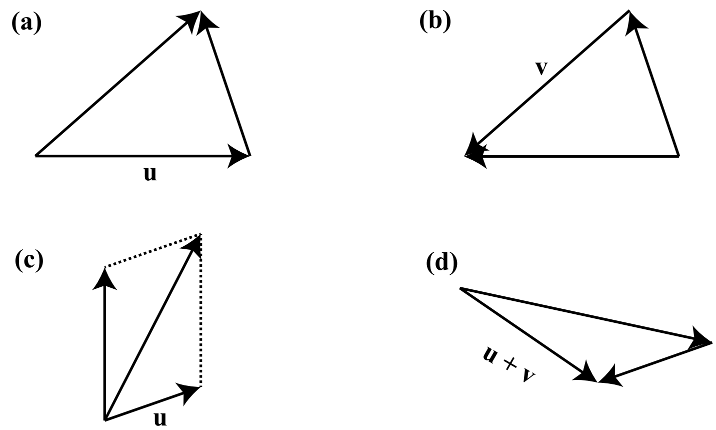
The vector $\mathbf{v} + \mathbf{w}$ is called the **sum** or **resultant** of the vectors $\mathbf{v}$ and $\mathbf{w}$.

The Angle Between Two Vectors

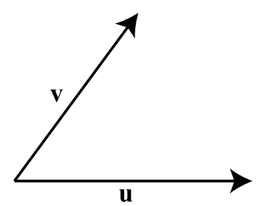
The angle θ between vectors is the angle formed by these two vectors (with $0° \le \theta \le 180°$) when they have the same initial point.

Exercises for Section 3.5

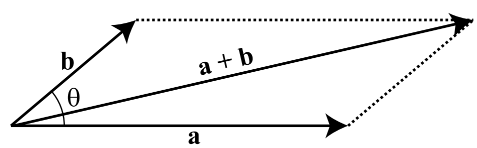
* **1.** In each of the following diagrams, one of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ is labeled. Label the other two vectors to make the diagram a valid representation of $\mathbf{u} + \mathbf{v}$.



★ **2.** On the following diagram, draw the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}+\mathbf{v}$, and $2\mathbf{u}-\mathbf{v}$.



★ **3.** In the following diagram, $|\mathbf{a}| = 10$ and $|\mathbf{a}+\mathbf{b}| = 14$. In addition, the angle θ between the vectors **a** and **b** is 30°. Determine the magnitude of the vector **b** and the angle between the vectors **a** and $\mathbf{a}+\mathbf{b}$.



4. Suppose that vectors **a** and **b** have magnitudes of 125 and 180, respectively. Also assume that the angle between these two vectors is 35°. Determine the magnitude of the vector $\mathbf{a}+\mathbf{b}$ and the measure of the angle between the vectors **a** and $\mathbf{a}+\mathbf{b}$.

5. A car that weighs 3250 pounds is on an inclined plane that makes an angle of 4.5° with the horizontal. Determine the magnitude of the force of the car on the inclined plane, and determine the magnitude of the force on the car down the plane due to gravity. What is the magnitude of the smallest force necessary to keep the car from rolling down the plane?

6. An experiment determined that a force of 45 pounds is necessary to keep a 250 pound object from sliding down an inclined plane. Determine the angle the inclined plane makes with the horizontal.

7. A cable that can withstand a force of 4500 pounds is used to pull an object up an inclined plane that makes an angle of 15 degrees with the horizontal. What is the heaviest object that can be pulled up this plane with the cable? (Assume that friction can be ignored.)

3.6 Vectors from an Algebraic Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we find the component form of a vector?
- How do we find the magnitude and the direction of a vector written in component form?
- How do we add and subtract vectors written in component form and how do we find the scalar product of a vector written in component form?
- What is the dot product of two vectors?
- What does the dot product tell us about the angle between two vectors?
- How do we find the projection of one vector onto another?

Introduction and Terminology

We have seen that a vector is completely determined by magnitude and direction. So two vectors that have the same magnitude and direction are equal. That means that we can position our vector in the plane and identify it in different ways. For one, we can place the initial point of a vector **v** at the origin and the terminal point will wind up at some point (v_1, v_2) as illustrated in Figure 3.29.

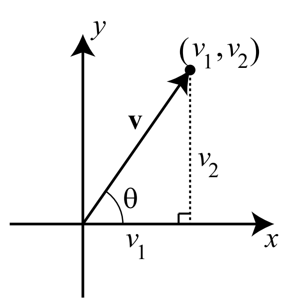


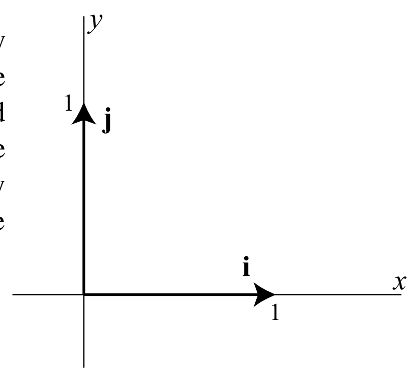
Figure 3.29: A Vector in Standard Position

A vector with its initial point at the origin is said to be in **standard position** and is represent by $\mathbf{v} = \langle v_1, v_2 \rangle$. Please note the important distinction in notation between the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ and the point (v_1, v_2). The coordinates of the terminal point (v_1, v_2) are called the **components** of the vector $\mathbf{v}$. We call $\mathbf{v} = \langle v_1, v_2 \rangle$ the **component form** of the vector $\mathbf{v}$. The first coordinate v_1 is called the x-component or the **horizontal component** of the vector $\mathbf{v}$, and the second coordinate v_2 is called the y-component or the **vertical component** of the vector $\mathbf{v}$. The nonnegative angle θ between the vector and the positive x-axis (with $0 \leq \theta < 360°$) is called the **direction angle** of the vector. See Figure 3.29.

Using Basis Vectors

There is another way to algebraically write a vector if the components of the vector are known. This uses the so-called **standard basis vectors** for vectors in the plane. These two vectors are denoted by **i** and **j** and are defined as follows and are shown in the diagram to the right.

$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle.$$



The diagram in Figure 3.30 shows how to use the vectors **i** and **j** to represent a vector $\mathbf{v} = \langle a, b \rangle$.

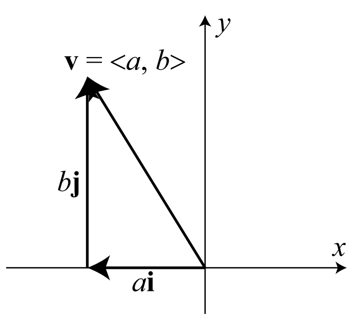


Figure 3.30: Using the Vectors **i** and **j**

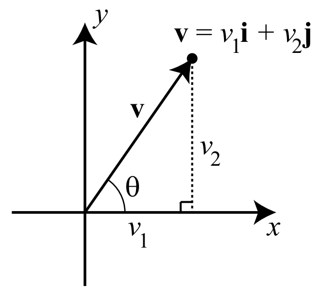
The diagram shows that if we place the tail of the vector $b\mathbf{j}$ at the tip of the vector $a\mathbf{i}$, we see that

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}.$$

This is often called the **i**, **j form of a vector**, and the real number a is called the **i-component** of **v** and the real number b is called the **j-component** of **v**

Algebraic Formulas for Geometric Properties of a Vector

Vectors have certain geometric properties such as length and a direction angle. With the use of the component form of a vector, we can write algebraic formulas for these properties. We will use the diagram to the right to help explain these formulas.



- The magnitude (or length) of the vector **v** is the distance from the origin to the point (v_1, v_2) and so

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}.$$

- The direction angle of **v** is θ, where $0 \leq \theta < 360°$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \quad \text{and} \quad \sin(\theta) = \frac{v_2}{|\mathbf{v}|}.$$

- The horizontal component and vertical component of the vector **v** with direction angle θ are

$$v_1 = |\mathbf{v}|\cos(\theta) \quad \text{and} \quad v_2 = |\mathbf{v}|\sin(\theta).$$

Progress Check 3.31 (Using the Formulas for a Vector)

1. Suppose the horizontal component of a vector **v** is 7 and the vertical component is -3. So we have $\mathbf{v} = 7\mathbf{i} + (-3)\mathbf{j} = 7\mathbf{i} - 3\mathbf{j}$. Determine the magnitude and the direction angle of **v**.

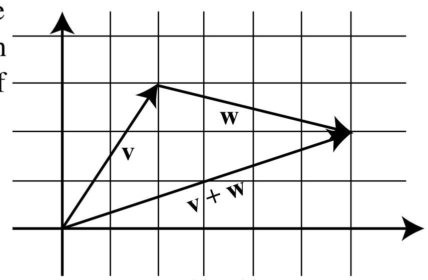
2. Suppose a vector **w** has a magnitude of 20 and a direction angle of 200°. Determine the horizontal and vertical components of **w** and write **w** in **i**, **j** form.

Operations on Vectors

In Section 3.5, we learned how to add two vectors and how to multiply a vector by a scalar. At that time, the descriptions of these operations was geometric in nature. We now know about the component form of a vector. So a good question is, "Can we use the component form of vectors to add vectors and multiply a vector by a scalar?"

To illustrate the idea, we will look at Progress Check 3.25 on page 223, where we added two vectors **v** and **w**. Although we did not use the component forms of these vectors, we can now see that

$$\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}, \quad \text{and}$$
$$\mathbf{w} = \langle 4, -1 \rangle = 4\mathbf{i} + (-1)\mathbf{j}.$$



The diagram to the right was part of the solutions for this progress check but now shows the vectors in a coordinate plane.

Notice that

$$\mathbf{v} + \mathbf{w} = 6\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{v} + \mathbf{w} = (2 + 4)\mathbf{i} + (3 + (-1))\mathbf{j}$$

Figure 3.31 shows a more general diagram with

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j} \;\text{ and }\; \mathbf{w} = \langle c, d \rangle = c\mathbf{i} + d\mathbf{j}$$

in standard position. This diagram shows that the terminal point of $\mathbf{v} + \mathbf{w}$ in standard position is $(a + c, b + d)$ and so

$$\mathbf{v} + \mathbf{w} = \langle a + c, b + d \rangle = (a + c)\mathbf{i} + (b + d)\mathbf{j}.$$

This means that we can add two vectors by adding their horizontal components and by adding their vertical components. The next progress check will illustrate something similar for scalar multiplication.

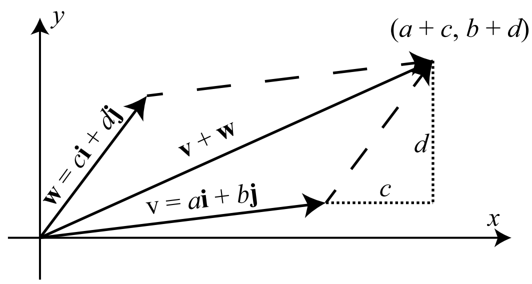


Figure 3.31: The Sum of Two Vectors

Progress Check 3.32 (Scalar Multiple of a Vector)

1. Let $\mathbf{v} = \langle 3, -2 \rangle$. Draw the vector $\mathbf{v}$ in standard position and then draw the vectors $2\mathbf{v}$ and $-2\mathbf{v}$ in standard position. What are the component forms of the vectors $2\mathbf{v}$ and $-2\mathbf{v}$?
2. In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ by a scalar c should be defined? Write a formal definition of a scalar multiple of a vector based on your intuition.

Based on the work we have done, we make the following formal definitions.

Definition. For vectors $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2 \rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and scalar c, we make the following definitions:

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle v_1 + w_1, v_2 + w_2 \rangle & \mathbf{v} + \mathbf{w} &= (v_1 + w_1)\,\mathbf{i} + (v_2 + w_2)\,\mathbf{j} \\ \mathbf{v} - \mathbf{w} &= \langle v_1 - w_1, v_2 - w_2 \rangle & \mathbf{v} - \mathbf{w} &= (v_1 - w_1)\,\mathbf{i} + (v_2 - w_2)\,\mathbf{j} \\ c\mathbf{v} &= \langle cv_1, cv_2 \rangle & c\mathbf{v} &= (cv_1)\,\mathbf{i} + (cv_2)\,\mathbf{j} \end{aligned}$$

Progress Check 3.33 (Vector Operations
Let $\mathbf{u} = \langle 1, -2 \rangle$, $\mathbf{v} = \langle 0, 4 \rangle$, and $\mathbf{w} = \langle -5, 7 \rangle$.

1. Determine the component form of the vector $2\mathbf{u} - 3\mathbf{v}$.
2. Determine the magnitude and the direction angle for $2\mathbf{u} - 3\mathbf{v}$.
3. Determine the component form of the vector $\mathbf{u} + 2\mathbf{v} - 7\mathbf{w}$.

The Dot Product of Two Vectors

Finding optimal solutions to systems is an important problem in applied mathematics. It is often the case that we cannot find an exact solution that satisfies certain constraints, so we look instead for the "best" solution that satisfies the constraints. An example of this is fitting a least squares curve to a set of data like our calculators do when computing a sine regression curve. The dot product is useful in these situations to find "best" solutions to certain types of problems. Although we won't see it in this course, having collections of perpendicular vectors is very important in that it allows for fast and efficient computations. The dot product of vectors allows us to measure the angle between them and thus determine if the vectors are perpendicular. The dot product has many applications, e.g., finding components of forces acting in different directions in physics and engineering. We introduce and investigate dot products in this section.

We have seen how to add vectors and multiply vectors by scalars, but we have not yet introduced a product of vectors. In general, a product of vectors should give us another vector, but there turns out to be no really useful way to define such a product of vectors. However, there is a dot "product" of vectors whose output is a scalar instead of a vector, and the dot product is a very useful product (even though it isn't a product of vectors in a technical sense).

Recall that the magnitude (or length) of the vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{u_1 u_1 + u_2 u_2}. \tag{1}$$

The expression under the second square root is a special case of important number we call the dot product of two vectors.

Definition. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors in the plane. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

So we see that for $\mathbf{u} = \langle u_1, u_2 \rangle$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= u_1 u_1 + u_2 u_2 = u_1^2 + u_2^2 \\ \mathbf{u} \cdot \mathbf{u} &= |\mathbf{u}|^2 \end{aligned} \tag{2}$$

So the dot product is related to the length of a vector, and it turns out that the dot product of two vectors is also useful in determining the angle between two vectors.

Recall that in Progress Check 3.27 on page 225, we used the Law of Cosines to determine the sum of two vectors and then used the Law of Sines to determine the angle between the sum and one of those vectors. We have now seen how much easier it is to compute the sum of two vectors when the vectors are in component form. The dot product will allow us to determine the cosine of the angle between two vectors in component form. This is due to the following result:

The Dot Product and the Angle between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v** ($0° \leq \theta \leq 180°$) as shown in Figure 3.35, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta) \qquad \text{or} \qquad \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Notice that if we have written the vectors **u** and **v** in component form, then we have formulas to compute $|\mathbf{u}|$, $|\mathbf{v}|$, and $\mathbf{u} \cdot \mathbf{v}$. This result may seem surprising but it is a fairly direct consequence of the Law of Cosines. This will be shown in the appendix at the end of this section.

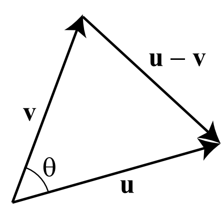


Figure 3.32: The angle between **u** and **v**

Progress Check 3.34 (Using the Dot Product)

1. Determine the angle θ between the vectors $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$.
2. Determine all vectors perpendicular to $\mathbf{u} = \langle 1, 3 \rangle$. How many such vectors are there? **Hint**: Let $\mathbf{v} = \langle a, b \rangle$. Under what conditions will the angle between **u** and **v** be 90°?

One purpose of Progress Check 3.34 was to use the formula

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

to determine when two vectors are perpendicular. Two vectors **u** and **v** will be perpendicular if and only if the angle θ between them is 90°. Since $\cos(90°) = 0$, we see that this formula implies that **u** and **v** will be perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. (This is because a fraction will be equal to 0 only when the numerator is equal to 0 and the denominator is not zero.) So we have

Two vectors are perpendicular if and only if their dot product is equal to 0.

Note: When two vectors are perpendicular, we also say that they are **orthogonal**.

Projections

Another useful application of the dot product is in finding the projection of one vector onto another. An example of where such a calculation is useful is the following.

> Usain Bolt from Jamaica excited the world of track and field in 2008 with his world record performances on the track. Bolt won the 100 meter race in a world record time of 9.69 seconds. He has since bettered that time with a race of 9.58 seconds with a wind assistance of 0.9 meters per second in Berlin on August 16, 2009 The wind assistance is a measure of the wind speed that is helping push the runners down the track. It is much easier to run a very fast race if the wind is blowing hard in the direction of the race. So that world records aren't dependent on the weather conditions, times are only recorded as record times if the wind aiding the runners is less than or equal to 2 meters per second. Wind speed for a race is recorded by a wind gauge that is set up close to the track. It is important to note, however, that weather is not always as cooperative as we might like. The wind does not always blow exactly in the direction of the track, so the gauge must account for the angle the wind makes with the track.

If the wind is blowing in the direction of the vector **u** and the track is in the direction of the vector **v** in Figure 3.33, then only part of the total wind vector is actually working to help the runners. This part is called the **projection of the vector u onto the vector v** and is denoted $\mathbf{proj_v u}$.

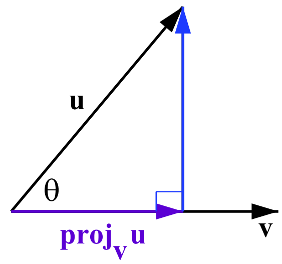


Figure 3.33: The Projection of **u** onto **v**

We can find this projection with a little trigonometry. To do so, we let θ be the angle between **u** and **v** as shown in Figure 3.33. Using right triangle trigonometry, we see that

$$\begin{aligned}|\mathbf{proj_v u}| &= |\mathbf{u}|\cos(\theta)\\ &= |\mathbf{u}|\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|}\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.\end{aligned}$$

The quantity we just derived is called the **scalar projection (or component) of u onto v** and is denoted by $\mathbf{comp_v u}$. Thus

$$\mathbf{comp_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

This gives us the length of the vector projection. So to determine the vector, we use a scalar multiple of this length times a unit vector in the same direction, which is $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$. So we obtain

$$\begin{aligned}\mathbf{proj_v u} &= |\mathbf{proj_v u}|\left(\frac{1}{|\mathbf{v}|}\mathbf{v}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\left(\frac{1}{|\mathbf{v}|}\mathbf{v}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\end{aligned}$$

We can also write the projection of **u** onto **v** as

$$\mathbf{proj_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

The wind component that acts perpendicular to the direction of $\mathbf{v}$ in Figure 3.33 is called the **projection of u orthogonal to v** and is denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$ as shown in Figure 3.34. Since $\mathbf{u} = \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\mathbf{v}}\mathbf{u}$, we have that

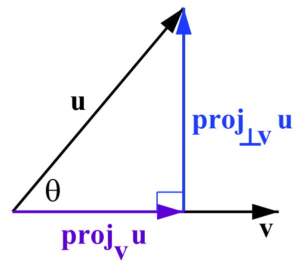


Figure 3.34: The Projection of **u** onto **v**

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}.$$

Following is a summary of the results we have obtained.

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, the **projection of the vector u onto the vector v**, $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$, is given by

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

See Figure 3.34. The **projection of u orthogonal to v**, denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, is

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}.$$

We note that $\mathbf{u} = \mathbf{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$.

Progress Check 3.35 (Projection of One Vector onto Another Vector

Let $\mathbf{u} = \langle 7, 5\rangle$ and $\mathbf{v} = \langle 10, -2\rangle$. Determine $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$, $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, and verify that $\mathbf{u} = \mathbf{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$. Draw a picture showing all of the vectors involved in this.

Appendix – Properties of the Dot Product

The main purpose of this appendix is to provide a proof of the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors. To do this, we first need to establish some properties of the dot product. The following shows the properties we will be using.

Let **a**, **b**, and **c** be vectors. Then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (Commutative Property).
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$ (Distributive Properties).

We have already established the first property on page 237. To prove the other results, we use $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$. We will also use the commutative property and distributive property for real numbers.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 \\ &= b_1a_1 + b_2a_2 \\ &= \mathbf{b}\cdot\mathbf{a}\end{aligned}$$

This establishes the commutative property for the dot product. For the distributive property, we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \langle a_1, a_2 \rangle\cdot\left(\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle\right) \\ &= \langle a_1, a_2 \rangle\cdot\langle b_1 + c_1, b_2 + c_2 \rangle \\ &= a_1\left(b_1 + c_1\right) + a_2\left(b_2 + c_2\right) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2\end{aligned}$$

We now rearrange the terms on the right side of the last equation to obtain

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= (a_1b_1 + a_2b_2) + (a_1c_1 + a_2c_2) \\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}\end{aligned}$$

This establishes one of the distributive properties. The other is proven in a similar manner.

We are now able to provide a proof of the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors. Let θ be the angle between the nonzero vectors **u** and **v** as illustrated in Figure 3.35.

We can apply the Law of Cosines to using the angle θ as follows:

$$\begin{aligned}|\mathbf{u}-\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta) \\ (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta) \qquad (3)\end{aligned}$$

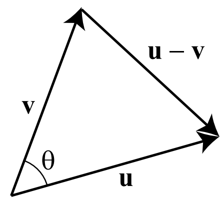


Figure 3.35: The angle between **u** and **v**

We now apply some of the properties of the dot product to the left side of equation (3).

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}-\mathbf{v})\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 \end{aligned} \qquad (4)$$

We can now use equations (3) and (4) to obtain

$$\begin{aligned}|\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta) \\ -2(\mathbf{u}\cdot\mathbf{v}) &= -2\,|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta) \\ \mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta).\end{aligned}$$

This is the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors.

Summary of Section 3.6

In this section, we studied the following important concepts and ideas:
The **component form** of a vector **v** is written as $\mathbf{v} = \langle v_1, v_2\rangle$ and the **i**, **j** form of the same vector is $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. Using this notation, we have

- The magnitude (or length) of the vector **v** is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.
- The direction angle of **v** is θ, where $0 \le \theta < 360°$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \quad \text{and} \quad \sin(\theta) = \frac{v_2}{|\mathbf{v}|}.$$

- The horizontal and component and vertical component of the vector **v** and direction angle θ are

$$v_1 = |\mathbf{v}|\cos(\theta) \quad \text{and} \quad v_2 = |\mathbf{v}|\sin(\theta).$$

For two vectors **v** and **w** with $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2 \rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and a scalar c:

- $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle = (v_1 + w_1)\,\mathbf{i} + (v_2 + w_2)\,\mathbf{j}$.
- $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle = (v_1 - w_1)\,\mathbf{i} + (v_2 - w_2)\,\mathbf{j}$.
- $c\mathbf{v} = \langle cv_1, cv_2 \rangle = (cv_1)\,\mathbf{i} + (cv_2)\,\mathbf{j}$.
- The **dot product** of **v** and **w** is $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2$.
- If θ is the angle between **v** and **w**, then

$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta) \qquad \text{or} \qquad \cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}.$$

- The **projection of the vector v onto the vector w**, $\mathbf{proj}_{\mathbf{w}}\mathbf{v}$, is given by

$$\mathbf{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}.$$

  The **projection of v orthogonal to w**, denoted $\mathbf{proj}_{\perp\mathbf{w}}\mathbf{v}$, is

$$\mathbf{proj}_{\perp\mathbf{w}}\mathbf{v} = \mathbf{v} - \mathbf{proj}_{\mathbf{w}}\mathbf{v}.$$

  We note that $\mathbf{v} = \mathbf{proj}_{\mathbf{w}}\mathbf{v} + \mathbf{proj}_{\perp\mathbf{w}}\mathbf{v}$. See Figure 3.36.

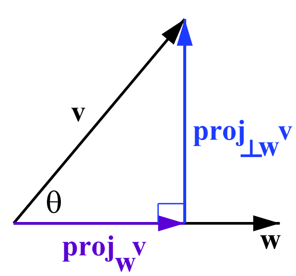


Figure 3.36: The Projection of **v** onto **w**

Exercises for Section 3.6

1. Determine the magnitude and the direction angle of each of the following vectors.

⋆ **(a)** $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

⋆ **(b)** $\mathbf{w} = \langle -3, 6 \rangle$

(c) $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$

(d) $\mathbf{u} = \langle -3, -5 \rangle$

2. Determine the horizontal and vertical components of each of the following vectors. Write each vector in $\mathbf{i}, \mathbf{j}$ form.

⋆ **(a)** The vector $\mathbf{v}$ with magnitude 12 and direction angle 50°.

⋆ **(b)** The vector $\mathbf{u}$ with $|\mathbf{u}| = \sqrt{20}$ and direction angle 125°.

(c) The vector $\mathbf{w}$ with magnitude 5.25 and direction angle 200°.

3. Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$, and $\mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$. Determine the $\mathbf{i}, \mathbf{j}$ form of each of the following:

⋆ **(a)** $5\mathbf{u} - \mathbf{v}$

(b) $2\mathbf{v} + 7\mathbf{w}$

⋆ **(c)** $\mathbf{u} + \mathbf{v} + \mathbf{w}$

(d) $3\mathbf{u} + 5\mathbf{w}$

4. Determine the value of the dot product for each of the following pairs of vectors.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

⋆ **(b)** $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 30°.

(c) $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 150°.

(d) $\mathbf{a}$ and $\mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is 50°.

(e) $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$.

5. Determine the angle between each of the following pairs of vectors.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

(b) $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$.

(c) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j}$.

6. For each pair of vectors, determine $\mathbf{proj_v w}$, $\mathbf{proj_{\perp v} w}$, and verify that $\mathbf{w} = \mathbf{proj_v w} + \mathbf{proj_{\perp v} w}$. Draw a picture showing all of the vectors involved in this.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

(b) $\mathbf{v} = \langle -2, 3 \rangle$ and $\mathbf{w} = \langle 1, 1 \rangle$

Chapter 4

Trigonometric Identities and Equations

Trigonometric identities describe equalities between related trigonometric expressions while trigonometric equations ask us to determine the specific values of the variables that make two expressions equal. Identities are tools that can be used to simplify complicated trigonometric expressions or solve trigonometric equations. In this chapter we will prove trigonometric identities and derive the double and half angle identities and sum and difference identities. We also develop methods for solving trigonometric equations, and learn how to use trigonometric identities to solve trigonometric equations.

4.1 Trigonometric Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is an identity?
- How do we verify an identity?

Consider the trigonometric equation $\sin(2x) = \cos(x)$. Based on our current knowledge, an equation like this can be difficult to solve exactly because the periods of the functions involved are different. What will allow us to solve this equation relatively easily is a trigonometric identity, and we will explicitly solve this equation in a subsequent section. This section is an introduction to trigonometric identities.

As we discussed in Section 2.6, a mathematical **equation** like $x^2 = 1$ is a relation between two expressions that may be true for some values of the variable. To solve an equation means to find all of the values for the variables that make the two expressions equal to each other. An **identity**, is an equation that is true for all allowable values of the variable. For example, from previous algebra courses, we have seen that

$$x^2 - 1 = (x + 1)(x - 1),$$

for all real numbers x. This is an algebraic identity since it is true for all real number values of x. An example of a trigonometric identity is

$$\cos^2(x) + \sin^2(x) = 1$$

since this is true for all real number values of x.

So while we solve equations to determine when the equality is valid, there is no reason to solve an identity since the equality in an identity is always valid. Every identity is an equation, but not every equation is an identity. To know that an equation is an identity it is necessary to provide a convincing argument that the two expressions in the equation are always equal to each other. Such a convincing argument is called a *proof* and we use proofs to verify trigonometric identities.

Definition. An **identity** is an equation that is true for all allowable values of the variables involved.

Beginning Activity

1. Use a graphing utility to draw the graph of $y = \cos\left(x - \dfrac{\pi}{2}\right)$ and $y = \sin\left(x + \dfrac{\pi}{2}\right)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Are the two expressions $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin\left(x + \dfrac{\pi}{2}\right)$ the same – that is, do they have the same value for every input x? If so, explain how the graphs indicate that the expressions are the same. If not, find at least one value of x at which $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin\left(x + \dfrac{\pi}{2}\right)$ have different values.

2. Use a graphing utility to draw the graph of $y = \cos\left(x - \dfrac{\pi}{2}\right)$ and $y = \sin(x)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Are the two expressions $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin(x)$ the same – that is, do they have the same value for every input x? If so, explain how the graphs indicate that the expressions are the same. If not, find at least one value of x at which $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin(x)$ have different values.

Some Known Trigonometric Identities

We have already established some important trigonometric identities. We can use the following identities to help establish new identities.

The Pythagorean Identity

This identity is fundamental to the development of trigonometry. See page 18 in Section 1.2.

$$\text{For all real numbers } t, \cos^2(t) + \sin^2(t) = 1.$$

Identities from Definitions

The definitions of the tangent, cotangent, secant, and cosecant functions were introduced in Section 1.6. The following are valid for all values of t for which the right side of each equation is defined.

$$\tan(t) = \frac{\sin(t)}{\cos(t)} \qquad \cot(t) = \frac{\cos(t)}{\sin(t)}$$
$$\sec(t) = \frac{1}{\cos(t)} \qquad \csc(t) = \frac{1}{\sin(t)}$$

Negative Identities

The negative were introduced in Chapter 2 when the symmetry of the graphs were discussed. (See page 82 and Exercise (2) on page 139.)

$$\cos(-t) = \cos(t) \qquad \sin(-t) = -\sin(t) \qquad \tan(-t) = -\tan(t).$$

The negative identities for cosine and sine are valid for all real numbers t, and the negative identity for tangent is valid for all real numbers t for which $\tan(t)$ is defined.

Verifying Identities

Given two expressions, say $\tan^2(x) + 1$ and $\sec^2(x)$, we would like to know if they are equal (that is, have the same values for every allowable input) or not. We can draw the graphs of $y = \tan^2(x) + 1$ and $y = \sec^2(x)$ and see if the graphs look the same or different. Even if the graphs look the same, as they do with $y = \tan^2(x)+1$ and $y = \sec^2(x)$, that is only an indication that the two expressions are equal for *every* allowable input. In order to verify that the expressions are in fact always equal, we need to provide a convincing argument that works for all possible input. To do so, we use facts that we know (existing identities) to show that two trigonometric expressions are always equal. As an example, we will verify that the equation

$$\tan^2(x) + 1 = \sec^2(x) \tag{1}$$

is an identity.

A proper format for this kind of argument is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. It usually makes life easier to begin with the more complicated looking side (if there is one). In our example of equation (1) we might begin with the expression $\tan^2(x) + 1$.

Example 4.1 (Verifying a Trigonometric Identity)
To verify that equation (1) is an identity, we work with the expression $\tan^2(x) + 1$. It can often be a good idea to write all of the trigonometric functions in terms of the cosine and sine to start. In this case, we know that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, so we could begin by making this substitution to obtain the identity

$$\tan^2(x) + 1 = \left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1. \tag{2}$$

Note that this is an identity and so is valid for all allowable values of the variable. Next we can apply the square to both the numerator and denominator of the right hand side of our identity (2).

$$\left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 = \frac{\sin^2(x)}{\cos^2(x)} + 1. \tag{3}$$

Next we can perform some algebra to combine the two fractions on the right hand side of the identity (3) and obtain the new identity

$$\frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}. \tag{4}$$

Now we can recognize the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, which makes the right side of identity (4)

$$\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}. \tag{5}$$

Recall that our goal is to verify identity (1), so we need to transform the expression into $\sec^2(x)$. Recall that $\sec(x) = \dfrac{1}{\cos(x)}$, and so the right side of identity (5) leads to the new identity

$$\frac{1}{\cos^2(x)} = \sec^2(x),$$

which verifies the identity.

An argument like the one we just gave that shows that an equation is an identity is called a *proof*. We usually leave out most of the explanatory steps (the steps should be evident from the equations) and write a proof in one long string of identities as

$$\begin{aligned}
\tan^2(x) + 1 &= \left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 \\
&= \frac{\sin^2(x)}{\cos^2(x)} + 1 \\
&= \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}$$

To prove an identity is to show that the expressions on each side of the equation are the same for every allowable input. We illustrated this process with the equation $\tan^2(x) + 1 = \sec^2(x)$. To show that an equation isn't an identity it is enough to demonstrate that the two sides of the equation have different values at one input.

Example 4.2 (Showing that an Equation is not an Identity)
Consider the equation $\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$ that we encountered in the Beginning Activity. Although you can check that $\cos\left(x - \frac{\pi}{2}\right)$ and $\sin\left(x + \frac{\pi}{2}\right)$ are equal at some values, $\frac{\pi}{4}$ for example, they are not equal at all values – $\cos\left(0 - \frac{\pi}{2}\right) = 0$ but $\sin\left(0 + \frac{\pi}{2}\right) = 1$. Since an identity must provide an equality for *all* allowable

values of the variable, if the two expressions differ at one input, then the equation is not an identity. So the equation $\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$ is not an identity.

Example 4.2 illustrates an important point. to show that an equation is not an identity, it is enough to find one input at which the two sides of the equation are not equal. We summarize our work with identities as follows.

- To prove that an equation is an identity, we need to apply known identities to show that one side of the equation can be transformed into the other.
- To prove that an equation is not an identity, we need to find one input at which the two sides of the equation have different values.

Important Note: When proving an identity it might be tempting to start working with the equation itself and manipulate both sides until you arrive at something you know to be true. DO NOT DO THIS! By working with both sides of the equation, we are making the assumption that the equation is an identity – but this assumes the very thing we need to show. So the proper format for a proof of a trigonometric identity is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. It usually makes life easier to begin with the more complicated looking side (if there is one).

Example 4.3 (Verifying an Identity)
Consider the equation

$$2\cos^2(x) - 1 = \cos^2(x) - \sin^2(x).$$

Graphs of both sides appear to indicate that this equation is an identity. To prove the identity, we start with the left hand side:

$$\begin{aligned} 2\cos^2(x) - 1 &= \cos^2(x) + \cos^2(x) - 1 \\ &= \cos^2(x) + (1 - \sin^2(x)) - 1 \\ &= \cos^2(x) - \sin^2(x). \end{aligned}$$

Notice that in our proof we rewrote the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$ as $\cos^2(x) = 1 - \sin^2(x)$. Any proper rearrangement of an identity is also an identity, so we can manipulate known identities to use in our proofs as well.

To reiterate, the proper format for a proof of a trigonometric identity is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. There are no hard and fast methods for proving identities – it is a bit of an art. You must practice to become good at it.

Progress Check 4.4 (Verifying Identities)
For each of the following, use a graphing utility to graph both sides of the equation. If the graphs indicate that the equation is not an identity, find one value of x at which the two sides of the equation have different values. If the graphs indicate that the equation is an identity, prove the identity.

1. $\dfrac{\sec^2(x) - 1}{\sec^2(x)} = \sin^2(x)$

2. $\cos(x)\sin(x) = 2\sin(x)$

Summary of Section 4.1

In this section, we studied the following important concepts and ideas:
An **identity** is an equation that is true for all allowable values of the variables involved.

- To prove that an equation is an identity, we need to apply known identities to show that one side of the equation can be transformed into the other.
- To prove that an equation is not an identity, we need to find one input at which the two sides of the equation have different values.

Exercises for Section 4.1

1. For each of the following, use a graphing utility to graph each side of the given equation. If the equation appears to be an identity, prove the identity. If the equation appears to not be an identity, demonstrate one input at which the two sides of the equation have different values. Remember that when proving an identity, work to transform one side of the equation into the other using known identities. Some general guidelines are

I. Begin with the more complicated side.

II. It is often helpful to use the definitions to rewrite all trigonometric functions in terms of the cosine and sine.

III. When appropriate, factor or combine terms. For example, $\sin^2(x) + \sin(x)$ can be written as $\sin(x)(\sin(x) + 1)$ and $\dfrac{1}{\sin(x)} + \dfrac{1}{\cos(x)}$ can be written as the single fraction $\dfrac{\cos(x) + \sin(x)}{\sin(x)\cos(x)}$ with a common denominator.

IV. As you transform one side of the equation, keep the other side of the equation in mind and use identities that involve terms that are on the other side. For example, to verify that $\tan^2(x) + 1 = \dfrac{1}{\cos^2(x)}$, start with $\tan^2(x) + 1$ and make use of identities that relate $\tan(x)$ to $\cos(x)$ as closely as possible.

⋆ **(a)** $\cos(x)\tan(x) = \sin(x)$

⋆ **(b)** $\dfrac{\cot(s)}{\csc(s)} = \cos(s)$

(c) $\dfrac{\tan(s)}{\sec(s)} = \sin(s)$

(d) $\cot^2(x) + 1 = \csc^2(x)$

⋆ **(e)** $\sec^2(x) + \csc^2(x) = 1$

(f) $\cot(t) + 1 = \csc(t)(\cos(t) + \sin(t))$

(g) $\tan^2(\theta)(1 + \cot^2(\theta)) = \dfrac{1}{1 - \sin^2(\theta)}$

(h) $\dfrac{1 - \sin^2(\beta)}{\cos(\beta)} = \sin(\beta)$

(i) $\dfrac{1 - \sin^2(\beta)}{\cos(\beta)} = \cos(\beta)$

(j) $\sin^2(x) + \tan^2(x) + \cos^2(x) = \sec^2(x)$.

2. A student claims that $\cos(\theta) + \sin(\theta) = 1$ is an identity because $\cos(0) + \sin(0) = 1 + 0 = 0$. How would you respond to this student?

3. If a trigonometric equation has one solution, then the periodicity of the trigonometric functions implies that the equation will have infinitely many solutions. Suppose we have a trigonometric equation for which both sides of the equation are equal at infinitely many different inputs. Must the equation be an identity? Explain your reasoning.

4.2 Trigonometric Equations

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a trigonometric equation?
- What does it mean to solve a trigonometric equation?
- How is a trigonometric equation different from a trigonometric identity?

We have already learned how to solve certain types of trigonometric equations. In Section 2.6 we learned how to use inverse trigonometric functions to solve trigonometric equations.

Beginning Activity

Refer back to the method from Section 2.6 to find all solutions to the equation $\sin(x) = 0.4$.

Trigonometric Equations

When a light ray from a point P reflects off a surface at a point R to illuminate a point Q as shown at left in Figure 4.1, the light makes two angles α and β with a perpendicular to the surface. The angle α is called the *angle of incidence* and the angle β is called the *angle of reflection*. The Law of Reflection states that when light is reflected off a surface, the angle of incidence equals the angle of reflection. What happens if the light travels through one medium (say air) from a point P, deflects into another medium (say water) to travel to a point Q? Think about what happens if you look at an object in a glass of water. See the diagram on the right in Figure 4.1. Again the light makes two angles α and β with a perpendicular to the surface. The angle α is called the *angle of incidence* and the angle β is called the *angle of refraction*. If light travels from air into water, the Law of Refraction says that

$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{c_a}{c_w} \tag{6}$$

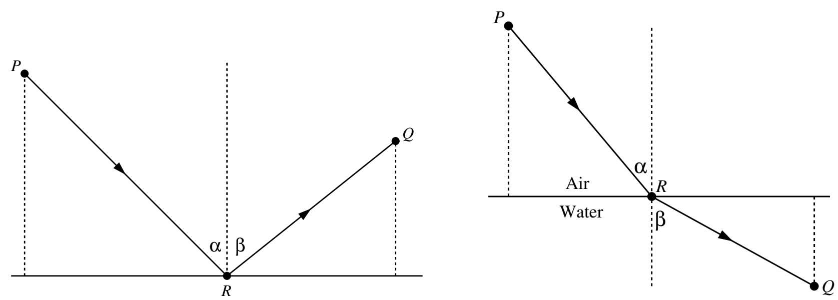


Figure 4.1: Reflection and refraction.

where c_a is the speed of light in air and c_w is the speed of light in water. The ratio $\dfrac{c_a}{c_w}$ of the speed of light in air to the speed of light in water can be calculated by experiment. In practice, the speed of light in each medium is compared to the speed of light in a vacuum. The ratio of the speed of light in a vacuum to the speed of light in water is around 1.33. This is called the index of refraction for water. The index of refraction for air is very close to 1, so the ratio $\dfrac{c_a}{c_w}$ is close to 1.33. We can usually measure the angle of incidence, so the Law of Refraction can tells us what the angle of refraction is by solving equation (6).

Trigonometric equations arise in a variety of situations, like in the Law of Refraction, and in a variety of disciplines including physics, chemistry, and engineering. As we develop trigonometric identities in this chapter, we will also use them to solve trigonometric equations.

Recall that Equation (6) is a *conditional* equation because it is not true for all allowable values of the variable. To *solve a conditional equation* means to find all of the values for the variables that make the two expressions on either side of the equation equal to each other.

Equations of Linear Type

Section 2.6 showed us how to solve trigonometric equations that are reducible to linear equations. We review that idea in our first example.

Example 4.5 (Solving an Equation of Linear Type)
Consider the equation

$$2\sin(x) = 1.$$

We want to find all values of x that satisfy this equation. Notice that this equation looks a lot like the linear equation $2y = 1$, with $\sin(x)$ in place of y. So this trigonometric equation is of linear type and we say that it is linear in $\sin(x)$. We know how to solve $2y = 1$, we simply divide both sides of the equation by 2 to obtain $y = \dfrac{1}{2}$. We can apply the same algebraic operation to $2\sin(x) = 1$ to obtain the equation

$$\sin(x) = \frac{1}{2}.$$

We could now proceed in a couple of ways. From previous work, we know that $\sin(x) = \dfrac{1}{2}$ when $x = \dfrac{\pi}{6}$. Alternatively, we could apply the inverse sine to both sides of our equation to see that one solution is $x = \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$.

Recall, however, this is not the only solution. The first task is to find all of the solutions in one complete period of the sine function. We can use the interval with $0 \le x \le 2\pi$ but we often use the interval $-\pi \le x \le \pi$. In this case, it makes no difference since the sine function is positive in the second quadrant. Using $\dfrac{\pi}{6}$ as a reference angle, we see that $x = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ is another solution of this equation. (Use a calculator to check this.)

We now use the fact that the sine function is periodic with a period of 2π to write formulas that can be used to generate all solutions of the equation $2\sin(x) = 1$. So the angles in the first quadrant are $\dfrac{\pi}{6} + k(2\pi)$ and the angles in the second quadrant are $\dfrac{5\pi}{6} + k(2\pi)$, where k is an integer. So for the solutions of the equation $2\sin(x) = 1$, we write

$$x = \frac{\pi}{6} + k(2\pi) \quad \text{or} \quad x = \frac{5\pi}{6} + k(2\pi),$$

where k is an integer.

We can always check our solutions by graphing both sides of the equation to see where the two expressions intersect. Figure 4.2 shows that graphs of $y = 2\sin(x)$ and $y = 1$ on the interval $[-2\pi, 3\pi]$. We can see that the points of intersection of these two curves occur at exactly the solutions we found for this equation.

Progress Check 4.6 (Solving an Equation of Linear Type)
Find the exact values of all solutions to the equation $4\cos(x) = 2\sqrt{2}$. Do this by first finding all solutions in one complete period of the cosine function and

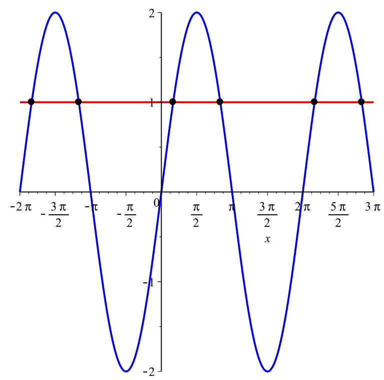


Figure 4.2: The graphs of $y = 2\sin(x)$ and $y = 1$

then using the periodic property to write formulas that can be used to generate all solutions of the equation. Draw appropriate graphs to illustrate your solutions.

Solving an Equation Using an Inverse Function

When we solved the equation $2\sin(x) = 1$, we used the fact that we know that $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$. When we cannot use one of the common arcs, we use the more general method of using an inverse trigonometric function. This is what we did in Section 2.6. See "A Strategy for Solving a Trigonometric Function" on page 158. We will illustrate this strategy with the equation $\cos(x) = 0.7$. We start by applying the inverse cosine function to both sides of this equation to obtain

$$\begin{aligned} \cos(x) &= 0.7 \\ \cos^{-1}(\cos(x) &= \cos^{-1}(0.7) \\ x &= \cos^{-1}(0.7) \end{aligned}$$

This gives the one solution for the equation that is in interval $[0, \pi]$. Before we use the periodic property, we need to determine the other solutions for the equation in one complete period of the cosine function. We can use the interval $[0, 2\pi]$ but it is easier to use the interval $[-\pi, \pi]$. One reason for this is the following so-called "negative arc identity" stated on page 82.

$$\cos(-x) = \cos(x) \quad \text{for every real number } x.$$

Hence, since one solution for the equation is $x = \cos^{-1}(0.7)$, another solution is $x = -\cos^{-1}(0.7)$. This means that the two solutions of the equation $x = \cos(x)$ on the interval $[-\pi, \pi]$ are

$$x = \cos^{-1}(0.7) \quad \text{and} \quad x = -\cos^{-1}(0.7).$$

Since the period of the cosine function is 2π, we can say that any solution of the equation $\cos(x) = 0.7$ will be of the form

$$x = \cos^{-1}(0.7) + k(2\pi) \quad \text{or} \quad x = -\cos^{-1}(0.7) + k(2\pi),$$

where k is some integer.

Note: The beginning activity for this section had the equation $\sin(x) = 0.4$. The solutions for this equation are

$$x = \arcsin(0.4) + k(2\pi) \text{ or } x = (\pi - \arcsin(0.4)) + k(2\pi),$$

where k is an integer. We can write the solutions in approximate form as

$$x = 0.41152 + k(2\pi) \text{ or } x = 2.73008 + k(2\pi),$$

where k is an integer.

Progress Check 4.7 (Solving Equations of Linear Type)

1. Determine formulas that can be used to generate all solutions to the equation $5\sin(x) = 2$. Draw appropriate graphs to illustrate your solutions in one period of the sine function.
2. Approximate, to two decimal places, the angle of refraction of light passing from air to water if the angle of incidence is $40°$. (Recall that the index of refraction for water is 1.33.)

Solving Trigonometric Equations Using Identities

We can use known trigonometric identities to help us solve certain types of trigonometric equations.

Example 4.8 (Using Identities to Solve Equations)
Consider the trigonometric equation

$$\cos^2(x) - \sin^2(x) = 1. \tag{7}$$

This equation is complicated by the fact that there are two different trigonometric functions involved. In this case, we can use the Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1$$

by solving for $\cos^2(x)$ to obtain

$$\cos^2(x) = 1 - \sin^2(x).$$

We can now substitute into equation (7) to get

$$(1 - \sin^2(x)) - \sin^2(x) = 1.$$

Note that everything is in terms of just the sine function and we can proceed to solve the equation from here:

$$\begin{aligned}
(1 - \sin^2(x)) - \sin^2(x) &= 1 \\
1 - 2\sin^2(x) &= 1 \\
-2\sin^2(x) &= 0 \\
\sin^2(x) &= 0 \\
\sin(x) &= 0.
\end{aligned}$$

We know that $\sin(x) = 0$ when $x = \pi k$ for any integer k, so the solutions to the equation

$$\cos^2(x) - \sin^2(x) = 1$$

are

$$x = \pi k \text{ for any integer } k.$$

This is illustrated by Figure 4.3.

Progress Check 4.9 (Using Identities to Solve Equations)
Find the exact values of all solutions to the equation $\sin^2(x) = 3\cos^2(x)$. Draw appropriate graphs to illustrate your solutions.

Other Methods for Solving Trigonometric Equations

Just like we did with linear equations, we can view some trigonometric equations as quadratic in nature and use tools from algebra to solve them.

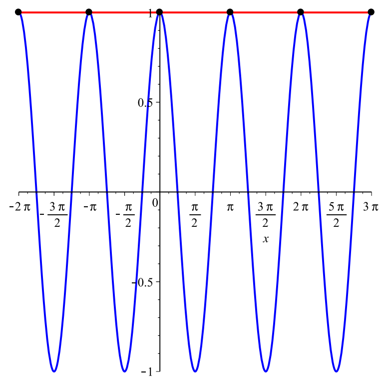


Figure 4.3: The graphs of $y = \cos^2(x) - \sin^2(x)$ and $y = 1$

Example 4.10 (Solving Trigonometric Equations of Quadratic Type)
Consider the trigonometric equation

$$\cos^2(x) - 2\cos(x) + 1 = 0.$$

This equation looks like a familiar quadratic equation $y^2 - 2y + 1 = 0$. We can solve this quadratic equation by factoring to obtain $(y-1)^2 = 0$. So we can apply the same technique to the trigonometric equation $\cos^2(x) - 2\cos(x) + 1 = 0$. Factoring the left hand side yields

$$(\cos(x) - 1)^2 = 0.$$

The only way $(\cos(x) - 1)^2$ can be 0 is if $\cos(x) - 1$ is 0. This reduces our quadratic trigonometric equation to a linear trigonometric equation. To summarize the process so far, we have

$$\begin{aligned} \cos^2(x) - 2\cos(x) + 1 &= 0 \\ (\cos(x) - 1)^2 &= 0 \\ \cos(x) - 1 &= 0 \\ \cos(x) &= 1. \end{aligned}$$

We know that $\cos(x) = 1$ when $x = 2\pi k$ for integer values of k. Therefore, the solutions to our original equation are

$$x = 2\pi k$$

where k is any integer. As a check, the graph of $y = \cos^2(x) - 2\cos(x) + 1$ is shown in Figure 4.4. The figure appears to show that the graph of $y = \cos^2(x) - 2\cos(x) + 1$ intersects the x-axis at exactly the points we found, so our solution is validated by graphical means.

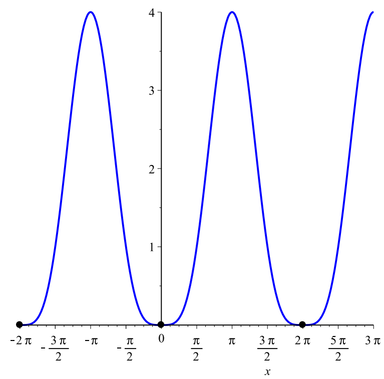


Figure 4.4: the graph of $y = \cos^2(x) - 2\cos(x) + 1$

Progress Check 4.11 (Solving Trigonometric Equations of Quadratic Type)
Find the exact values of all solutions to the equation $\sin^2(x) - 4\sin(x) = -3$. Draw appropriate graphs to illustrate your solutions.

Summary of Section 4.2

In this section, we studied the following important concepts and ideas:
A **trigonometric equation** is a conditional equation that involves trigonometric functions. If it is possible to write the equation in the form

$$\text{"some trigonometric function of } x\text{"} = \text{a number}, \tag{1}$$

we can use the following strategy to solve the equation:

- Find all solutions of the equation within one period of the function. This is often done by using properties of the trigonometric function. Quite often, there will be two solutions within a single period.

- Use the period of the function to express formulas for all solutions by adding integer multiples of the period to each solution found in the first step. For example, if the function has a period of 2π and x_1 and x_2 are the only two solutions in a complete period, then we would write the solutions for the equation as

$$x = x_1 + k(2\pi), \quad x = x_2 + k(2\pi), \text{ where } k \text{ is an integer.}$$

We can sometimes use trigonometric identities to help rewrite a given equation in the form of equation (1).

Exercises for Section 4.2

1. For each of the following equations, determine formulas that can be used to generate all solutions of the given equation. Use a graphing utility to graph each side of the given equation to check your solutions.

- ⋆ **(a)** $2\sin(x) - 1 = 0$
- ⋆ **(b)** $2\cos(x) + 1 = 0$
- **(c)** $2\sin(x) + \sqrt{2} = 0$
- ⋆ **(d)** $4\cos(x) - 3 = 0$
- **(e)** $3\sin^2(x) - 2\sin(x) = 0$
- **(f)** $\sin(x)\cos^2(x) = 2\sin(x)$
- **(g)** $\cos^2(x) + 4\sin(x) = 4$
- **(h)** $5\cos(x) + 4 = 2\sin^2(x)$
- **(i)** $3\tan^2(x) - 1 = 0$
- **(j)** $\tan^2(x) - \tan(x) = 6$

⋆ **2.** A student is asked to approximate all solutions in degrees (to two decimal places) to the equation $\sin(\theta) + \dfrac{1}{3} = 1$ on the interval $0° \leq \theta \leq 360°$. The student provides the answer $\theta = \sin^{-1}\left(\dfrac{2}{3}\right) \approx 41.81°$. Did the student provide the correct answer to the stated problem? Explain.

3. X-ray crystallography is an important tool in chemistry. One application of X-ray crystallography is to discover the atomic structure macromolecules. For example, the double helical structure of DNA was found using X-ray crystallography.

The basic idea behind X-ray crystallography is this: two X-ray beams with the same wavelength λ and phase are directed at an angle θ toward a crystal composed of atoms arranged in a lattice in planes separated by a distance d as illustrated in Figure 4.5.[1] The beams reflect off different atoms (labeled as P and Q in Figure 4.5) within the crystal. One X-ray beam (the lower one as

[1] The symbol λ is the Greek lowercase letter "lambda".

illustrated in Figure 4.5) must travel a longer distance than the other. When reflected, the X-rays combine but, because of the phase shift of the lower beam, the combination might have a small amplitude or a large amplitude. Bragg's Law states that the sum of the reflected rays will have maximum amplitude when the extra length the longer beam has to travel is equal to an integer multiple of the wavelength λ of the radiation. In other words,

$$n\lambda = 2d\sin(\theta),$$

for some positive integer n. Assume that $\lambda = 1.54$ angstroms and $d = 2.06$ angstroms. Approximate to two decimal places the smallest value of θ (in degrees) for which $n = 1$.

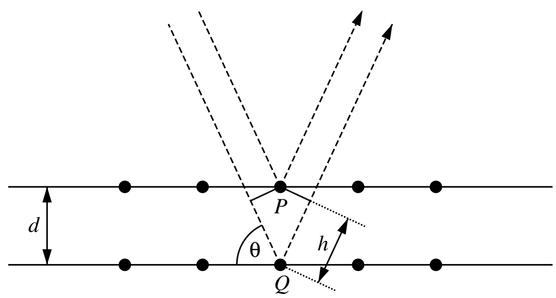


Figure 4.5: X-rays reflected from crystal atoms.

4.3 Sum and Difference Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Cosine Difference and Sum Identities?
- What are the Sine Difference and Sum Identities?
- What are the Tangent Difference and Sum Identities?
- What are the Cofunction Identities?
- Why are the difference and sum identities useful?

The next identities we will investigate are the sum and difference identities for the cosine and sine. These identities will help us find exact values for the trigonometric functions at many more angles and also provide a means to derive even more identities.

Beginning Activity

1. Is $\cos(A - B) = \cos(A) - \cos(B)$ an identity? Explain.

2. Is $\sin(A - B) = \sin(A) - \sin(B)$ an identity? Explain.

3. Use a graphing utility to draw the graph of $y = \sin\left(\frac{\pi}{2} - x\right)$ and $y = \cos(x)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Do you think $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$ is an identity? Why or why not?

The Cosine Difference Identity

Up to this point, we know the exact values of the trigonometric functions at only a few angles. Trigonometric identities can help us extend this list of angles at which we know exact values of the trigonometric functions. Consider, for example, the problem of finding the exact value of $\cos\left(\frac{\pi}{12}\right)$. The definitions and identities we have so far do not help us with this problem. However, we could notice that

$\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$ and if we knew how the cosine behaved with respect to the difference of two angles, then we could find $\cos\left(\dfrac{\pi}{12}\right)$. In our Beginning Activity, however, we saw that the equation $\cos(A - B) = \cos(A) - \cos(B)$ is not an identity, so we need to understand how to relate $\cos(A - B)$ to cosines and sines of A and B.

We state the Cosine Difference Identity below. This identity is not obvious, and a verification of the identity is given in the appendix for this section on page 271. For now we focus on using the identity.

Cosine Difference Identity
For any real numbers A and B we have

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B).$$

Example 4.12 (Using the Cosine Difference Identity)
Let us return to our problem of finding $\cos\left(\dfrac{\pi}{12}\right)$. Since we know $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$, we can use the Cosine Difference Identity with $A = \dfrac{\pi}{3}$ and $B = \dfrac{\pi}{4}$ to obtain

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}.
\end{aligned}$$

So we see that $\cos\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{2} + \sqrt{6}}{4}$. As a check on this work, we can use a calculator to verify that both sides of this equation are approximately 0.965926.

Progress Check 4.13 (Using the Cosine Difference Identity)

1. Determine the *exact* value of $\cos\left(\dfrac{7\pi}{12}\right)$ using the Cosine Difference Identity.

2. Given that $\dfrac{5\pi}{12} = \dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{\pi}{6} - \left(-\dfrac{\pi}{4}\right)$, determine the *exact* value of $\cos\left(\dfrac{5\pi}{12}\right)$ using the Cosine Difference Identity.

The Cosine Sum Identity

Since there is a Cosine Difference Identity, we might expect there to be a Cosine Sum Identity. We can use the Cosine Difference Identity along with the negative identities to find an identity for $\cos(A+B)$. The basic idea was contained in our last Progress Check, where we wrote $A+B$ as $A-(-B)$. To see how this works in general, notice that

$$\begin{aligned}\cos(A+B) &= \cos(A-(-B)) \\ &= \cos(A)\cos(-B)+\sin(A)\sin(-B) \\ &= \cos(A)\cos(B)-\sin(A)\sin(B).\end{aligned}$$

This is the Cosine Sum Identity.

Cosine Sum Identity
For any real numbers A and B we have

$$\cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B).$$

Progress Check 4.14 (Using the Cosine Sum and Difference Identities)

1. Find a simpler formula for $\cos(\pi+x)$ in terms of $\cos(x)$. Illustrate with a graph.
2. Use the Cosine Difference Identity to prove that $\cos\left(\frac{\pi}{2}-x\right) = \sin(x)$ is an identity.

Cofunction Identities

In Progress Check 4.14, we used the Cosine Difference Identity to see that $\cos\left(\frac{\pi}{2}-x\right) = \sin(x)$ is an identity. Since this is an identity, we can replace x with $\frac{\pi}{2}-x$ to see that

$$\sin\left(\frac{\pi}{2}-x\right) = \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right) = \cos(x),$$

so $\sin\left(\frac{\pi}{2}-x\right) = \cos(x)$. The two identities

$$\cos\left(\frac{\pi}{2}-x\right) = \sin(x) \quad \text{and} \quad \sin\left(\frac{\pi}{2}-x\right) = \cos(x)$$

are called *cofunction identities*. These two cofunction identities show that the sine and cosine of the acute angles in a right triangle are related in a particular way.

Since the sum of the measures of the angles in a right triangle is π radians or $180°$, the measures of the two acute angles in a right triangle sum to $\dfrac{\pi}{2}$ radians or $90°$. Such angles are said to be complementary. Thus, the sine of an acute angle in a right triangle is the same as the cosine of its complementary angle. For this reason we call the sine and cosine *cofunctions*. The naming of the six trigonometric functions reflects the fact that they come in three sets of cofunction pairs: the sine and cosine, the tangent and cotangent, and the secant and cosecant. The cofunction identities are the same for any cofunction pair.

Cofunction Identities

For any real number x for which the expressions are defined,

- $\cos\left(\dfrac{\pi}{2} - x\right) = \sin(x)$
- $\sin\left(\dfrac{\pi}{2} - x\right) = \cos(x)$
- $\tan\left(\dfrac{\pi}{2} - x\right) = \cot(x)$
- $\cot\left(\dfrac{\pi}{2} - x\right) = \tan(x)$
- $\sec\left(\dfrac{\pi}{2} - x\right) = \csc(x)$
- $\csc\left(\dfrac{\pi}{2} - x\right) = \sec(x)$

For any angle x in degrees for which the functions are defined,

- $\cos(90° - x) = \sin(x)$
- $\sin(90° - x) = \cos(x)$
- $\tan(90° - x) = \cot(x)$
- $\cot(90° - x) = \tan(x)$
- $\sec(90° - x) = \csc(x)$
- $\csc(90° - x) = \sec(x)$

Progress Check 4.15 (Using the Cofunction Identities)
Use the cosine and sine cofuntion identities to prove the cofunction identity

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x).$$

The Sine Difference and Sum Identities

We can now use the Cosine Difference Identity and the Cofunction Identities to derive a Sine Difference Identity:

$$\begin{aligned}\sin(A-B) &= \cos\left(\frac{\pi}{2}-(A-B)\right)\\ &= \cos\left(\left(\frac{\pi}{2}-A\right)+B)\right)\\ &= \cos\left(\frac{\pi}{2}-A\right)\cos(B)-\sin\left(\frac{\pi}{2}-A\right)\sin(B)\\ &= \sin(A)\cos(B)-\cos(A)\sin(B).\end{aligned}$$

We can derive a Sine Sum Identity from the Sine Difference Identity:

$$\begin{aligned}\sin(A+B) &= \sin(A-(-B))\\ &= \sin(A)\cos(-B)-\cos(A)\sin(-B)\\ &= \sin(A)\cos(B)+\cos(A)\sin(B).\end{aligned}$$

Sine Difference and Sum Identities
For any real numbers A and B we have

$$\sin(A-B) = \sin(A)\cos(B)-\cos(A)\sin(B)$$

and

$$\sin(A+B) = \sin(A)\cos(B)+\cos(A)\sin(B).$$

Progress Check 4.16 (Using the Sine Sum and Difference Identities)
Use the Sine Sum or Difference Identities to find the *exact* values of the following.

1. $\sin\left(\dfrac{\pi}{12}\right)$

2. $\sin\left(\dfrac{5\pi}{12}\right)$

Using Sum and Difference Identities to Solve Equations

As we have done before, we can use our new identities to solve other types of trigonometric equations.

Example 4.17 (Using the Cosine Sum Identity to Solve an Equation)
Consider the equation

$$\cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2}.$$

On the surface this equation looks quite complicated, but we can apply an identity to simplify it to the point where it can be solved. Notice that left side of this equation has the form $\cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = \theta$ and $B = \dfrac{\pi}{5}$. We can use the Cosine Sum Identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ to combine the terms on the left into a single term, and we can solve the equation from there:

$$\begin{aligned} \cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right) &= \frac{\sqrt{3}}{2} \\ \cos\left(\theta + \frac{\pi}{5}\right) &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Now $\cos(x) = \dfrac{\sqrt{3}}{2}$ when $x = \dfrac{\pi}{6} + 2k\pi$ or $x = -\dfrac{\pi}{6} + 2k\pi$ for integers k. Thus, $\cos\left(\theta + \dfrac{\pi}{5}\right) = \dfrac{\sqrt{3}}{2}$ when $\theta + \dfrac{\pi}{5} = \dfrac{\pi}{6} + 2k\pi$ or $\theta + \dfrac{\pi}{5} = -\dfrac{\pi}{6} + 2k\pi$. Solving for θ gives us the solutions

$$\theta = -\frac{\pi}{30} + 2k\pi \quad \text{or} \quad \theta = -\frac{11\pi}{30} + 2k\pi$$

where k is any integer. These solutions are illustrated in Figure 4.6.

Note: Up to now, we have been using the phrase "Determine formulas that can be used to generate all the solutions of a given equation." This is not standard terminology but was used to remind us of what we have to do to solve a trigonometric equation. We will now simply say, "Determine all solutions for the given equation." When we see this, we should realize that we have to determine formulas that can be used to generate all the solutions of a given equation.

Progress Check 4.18 (Using an Identity to Help Solve an Equation)
Determine all solutions of the equation

$$\sin(x)\cos(1) + \cos(x)\sin(1) = 0.2.$$

Hint: Use a sum or difference identity and use the inverse sine function.

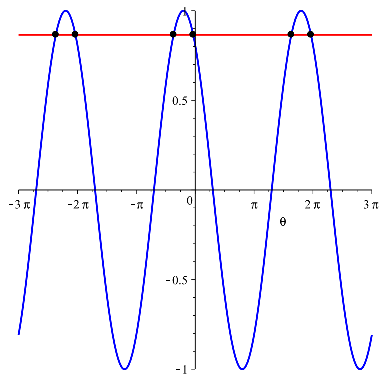


Figure 4.6: Graphs of $y = \cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right)$ and $y = \frac{\sqrt{3}}{2}$.

Appendix – Proof of the Cosine Difference Identity

To understand how to calculate the cosine of the difference of two angles, let A and B be arbitrary angles in radians. Figure 4.7 shows these angles with $A > B$, but the argument works in general. If we plot the points where the terminal sides of the angles A, B, and $A - B$ intersect the unit circle, we obtain the picture in Figure 4.7.

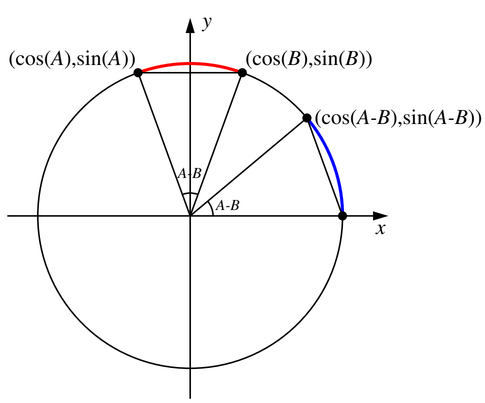


Figure 4.7: The cosine difference formula

The arc on the unit circle from the point $(\cos(B), \sin(B))$ to the point $(\cos(A), \sin(A))$ has length $A - B$, and the arc from the point (1,0) to the point $(\cos(A-B), \sin(A-B))$ also has length $A-B$. So the chord from $(\cos(B), \sin(B))$

to $(\cos(A), \sin(A))$ has the same length as the chord from (1,0) to $(\cos(A-B), \sin(A-B))$. To find the cosine difference formula, we calculate these two chord lengths using the distance formula.

The length of the chord from $(\cos(B), \sin(B))$ to $(\cos(A), \sin(A))$ is

$$\sqrt{(\cos(A)-\cos(B))^2+(\sin(A)-\sin(B))^2}$$

and the length of the chord from (1,0) to $(\cos(A-B), \sin(A-B))$ is

$$\sqrt{(\cos(A-B)-1)^2+(\sin(A-B)-0)^2}.$$

Since these two chord lengths are the same we obtain the equation

$$\begin{aligned}&\sqrt{(\cos(A-B)-1)^2+(\sin(A-B)-0)^2}\\&\qquad=\sqrt{(\cos(A)-\cos(B))^2+(\sin(A)-\sin(B))^2}. \quad (2)\end{aligned}$$

The cosine difference identity is found by simplifying Equation (2) by first squaring both sides:

$$\begin{aligned}&(\cos(A-B)-1)^2+(\sin(A-B)-0)^2\\&\qquad=(\cos(A)-\cos(B))^2+(\sin(A)-\sin(B))^2.\end{aligned}$$

Then we expand both sides

$$\begin{aligned}&[\cos^2(A-B)-2\cos(A-B)+1]+\sin^2(A-B)\\&=[\cos^2(A)-2\cos(A)\cos(B)+\cos^2(B)]+[\sin^2(A)-2\sin(A)\sin(B)+\sin^2(B)].\end{aligned}$$

We can combine some like terms:

$$\begin{aligned}&[\cos^2(A-B)+\sin^2(A-B)]-2\cos(A-B)+1\\&=[\cos^2(A)+\sin^2(A)]+[\cos^2(B)+\sin^2(B)]-2\cos(A)\cos(B)-2\sin(A)\sin(B).\end{aligned}$$

Finally, using the Pythagorean identities yields

$$\begin{aligned}1-2\cos(A-B)+1&=1+1-2\cos(A)\cos(B)-2\sin(A)\sin(B)\\-2\cos(A-B)&=-2\cos(A)\cos(B)-2\sin(A)\sin(B)\\\cos(A-B)&=\cos(A)\cos(B)+\sin(A)\sin(B).\end{aligned}$$

Summary of Section 4.3

In this section, we studied the following important concepts and ideas:

- **Sum and Difference Identities**

$$\begin{aligned}
\cos(A-B) &= \cos(A)\cos(B)+\sin(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B)-\cos(A)\sin(B)\\
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B)
\end{aligned}$$

- **Cofunction Identities**
  See page 268 for a list of the cofunction identities.

Exercises for Section 4.3

1. Use an appropriate sum or difference identity to find the exact value of each of the following.

⋆ **(a)** $\cos(-10°)\cos(35°)+\sin(-10°)\sin(35°)$

⋆ **(b)** $\cos\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)-\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

(c) $\sin\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

(d) $\sin(80°)\cos(55°)+\cos(80°)\sin(55°)$

2. Angles A and B are in standard position and $\sin(A)=\frac{1}{2}$, $\cos(A)>0$, $\cos(B)=\frac{3}{4}$, and $\sin(B)<0$. Draw a picture of the angles A and B in the plane and then find each of the following.

⋆ **(a)** $\cos(A+B)$

(b) $\cos(A-B)$

(c) $\sin(A+B)$

(d) $\sin(A-B)$

(e) $\tan(A+B)$

(f) $\tan(A-B)$

3. Identify angles A and B at which we know the values of the cosine and sine so that a sum or difference identity can be used to calculate the exact value of the given quantity. (For example, $15° = 45° - 30°$.)

- ⋆ **(a)** $\cos(15°)$
- **(b)** $\sin(75°)$
- **(c)** $\tan(105°)$
- ⋆ **(d)** $\sec(345°)$

4. Verify the sum and difference identities for the tangent:

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

and

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

.

5. Verify the cofunction identities

- ⋆ **(a)** $\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$
- **(b)** $\sec\left(\frac{\pi}{2} - x\right) = \csc(x)$
- **(c)** $\csc\left(\frac{\pi}{2} - x\right) = \sec(x)$

6. Draw graphs to determine if a given equation is an identity. Verify those equations that are identities and provide examples to show that the others are not identities.

- (**a**) $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 2\sin(x)\cos\left(\frac{\pi}{4}\right)$
- (**b**) $\sin(210° + x) - \cos(210° + x) = 0$

7. Determine if the following equations are identities.

- (**a**) $\dfrac{\sin(r + s)}{\cos(r)\cos(s)} = \tan(r) + \tan(s)$
- (**b**) $\dfrac{\sin(r - s)}{\cos(r)\cos(s)} = \tan(r) - \tan(s)$

8. Use an appropriate identity to solve the given equation.

- (**a**) $\sin(\theta)\cos(35°) + \cos(\theta)\sin(35°) = \frac{1}{2}$
- (**b**) $\cos(2x)\cos(x) + \sin(2x)\sin(x) = -1$

9. (**a**) Use a graphing device to draw the graph of $f(x) = \sin(x) + \cos(x)$ using $-\pi \le x \le 2\pi$ and $-2 \le y \le 2$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

(**b**) Use one of the sum identities to rewrite the expression $\sin\left(x + \dfrac{\pi}{4}\right)$. Then use the values of $\sin\left(\dfrac{\pi}{4}\right)$ and $\cos\left(\dfrac{\pi}{4}\right)$ to further rewrite the expression.

⋆ (**c**) Use the result from part (b) to show that the function $f(x) = \sin(x) + \cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

10. (**a**) Use a graphing device to draw the graph of $g(x) = \sin(x) + \sqrt{3}\cos(x)$ using $-\pi \le x \le 2\pi$ and $-2.5 \le y \le 2.5$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

(**b**) Use one of the sum identities to rewrite the expression $\sin\left(x + \dfrac{\pi}{3}\right)$. Then use the values of $\sin\left(\dfrac{\pi}{3}\right)$ and $\cos\left(\dfrac{\pi}{3}\right)$ to further rewrite the expression.

(**c**) Use the result from part (b) to show that the function $g(x) = \sin(x) + \sqrt{3}\cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

11. When two voltages are applied to a circuit, the resulting voltage in the circuit will be the sum of the individual voltages. Suppose two voltages $V_1(t) = 30\sin(120\pi t)$ and $V_2(t) = 40\cos(120\pi t)$ are applied to a circuit. The graph of the sum $V(t) = V_1(t) + V_2(t)$ is shown in Figure 4.8.

(**a**) Use the graph to estimate a value of C so that

$$y = 50\sin(120\pi(t - C))$$

fits the graph of V.

(**b**) Use the Sine Difference Identity to rewrite $50\sin(120\pi(t - C))$ as an expression of the form $50\sin(A)\cos(B) - 50\cos(A)\sin(B)$, where A and B involve t and/or C. From this, determine a value of C that will make

$$30\sin(120\pi t) + 40\cos(120\pi t) = 50\sin(120\pi(t - C)).$$

Compare this value of C to the one you estimated in part (a).

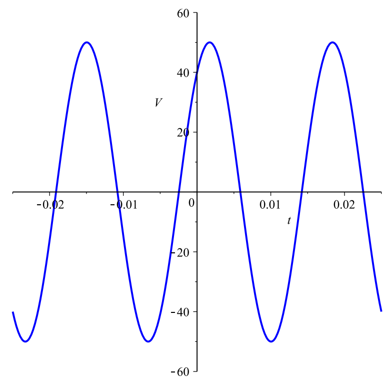


Figure 4.8: Graph of $V(t) = V_1(t) + V_2(t)$.

4.4 Double and Half Angle Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Double Angle Identities for the sine, cosine, and tangent?
- What are the Half Angle Identities for the sine, cosine, and tangent?
- What are the Product-to-Sum Identities for the sine and cosine?
- What are the Sum-to-Product Identities for the sine and cosine?
- Why are these identities useful?

The sum and difference identities can be used to derive the double and half angle identities as well as other identities, and we will see how in this section. Again, these identities allow us to determine exact values for the trigonometric functions at more points and also provide tools for solving trigonometric equations (as we will see later).

Beginning Activity

1. Use $B = A$ in the Cosine Sum Identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to write $\cos(2A)$ in terms of $\cos(A)$ and $\sin(A)$.

2. Is the equation

$$\frac{\cos(2x)}{2} = \cos(x)$$

an identity? Verify your answer.

The Double Angle Identities

Suppose a marksman is shooting a gun with muzzle velocity $v_0 = 1200$ feet per second at a target 1000 feet away. If we neglect all forces acting on the bullet except the force due to gravity, the horizontal distance the bullet will travel depends on the angle θ at which the gun is fired. If we let r be this horizontal distance (called the range), then

$$r = \frac{v_0^2}{g}\sin(2\theta),$$

where g is the gravitational force acting to pull the bullet downward. In this context, $g = 32$ feet per second per second, giving us

$$r = 45000\sin(2\theta).$$

The marksman would want to know the minimum angle at which he should fire in order to hit the target 1000 feet away. In other words, the marksman wants to determine the angle θ so that $r = 1000$. This leads to solving the equation

$$45000\sin(2\theta) = 1000. \tag{3}$$

Equations like the range equation in which multiples of angles arise frequently, and in this section we will determine formulas for $\cos(2A)$ and $\sin(2A)$ in terms of $\cos(A)$ and $\sin(A)$. These formulas are called *double angle identities*. In our Beginning Activity we found that

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

can be derived directly from the Cosine Sum Identity. A similar identity for the sine can be found using the Sine Sum Identity:

$$\begin{aligned}\sin(2A) &= \sin(A + A)\\ &= \sin(A)\cos(A) + \cos(A)\sin(A)\\ &= 2\cos(A)\sin(A).\end{aligned}$$

Progress Check 4.19 (Using the Double Angle Identities)
If $\cos(\theta) = \dfrac{5}{13}$ and $\dfrac{3\pi}{2} \le \theta \le 2\pi$, find $\cos(2\theta)$ and $\sin(2\theta)$.

There is also a double angle identity for the tangent. We leave the verification of that identity for the exercises. To summarize:

Double Angle Identities

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\sin(2A) = 2\cos(A)\sin(A)$$
$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)},$$

The first two identities are valid for all numbers A and the third is valid as long as $A \neq \frac{\pi}{4} + k\left(\frac{\pi}{2}\right)$, where k is an integer.

Progress Check 4.20 (Alternate Double Angle Identities)
Prove the alternate versions of the double angle identity for the cosine.

1. $\cos(2A) = 1 - 2\sin^2(A)$

2. $\cos(2A) = 2\cos^2(A) - 1$.

Solving Equations with Double Angles

Solving equations, like $45000\sin(2\theta) = 1000$, that involve multiples of angles, requires the same kind of techniques as solving other equations, but the multiple angle can add another wrinkle.

Example 4.21 (Solving an Equation with a Multiple Angle)
Consider the equation

$$2\cos(2\theta) - 1 = 0.$$

This is an equation that is linear in $\cos(2\theta)$, so we can apply the same ideas as we did earlier to this equation. We solve for $\cos(2\theta)$ to see that

$$\cos(2\theta) = \frac{1}{2}.$$

We know the angles at which the cosine has the value $\frac{1}{2}$, namely $\frac{\pi}{3} + 2\pi k$ and $-\frac{\pi}{3} + 2\pi k$ for integers k. In our case, this make

$$2\theta = \frac{\pi}{3} + 2\pi k \text{ or } 2\theta = -\frac{\pi}{3} + 2\pi k$$

for integers k. Now we divide both sides of these equations by 2 to find our solutions

$$\theta = \frac{\pi}{6} + \pi k \ \text{ or } \ \theta = -\frac{\pi}{6} + \pi k$$

for integers k. These solutions are illustrated in Figure 4.9.

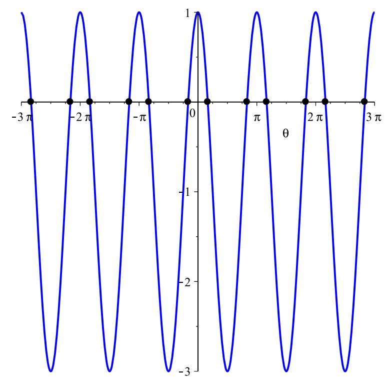


Figure 4.9: Graphs of $y = 2\cos(2\theta) - 1$.

Progress Check 4.22 (Solving Equations with Double Angles)
Approximate the smallest positive solution in degrees, to two decimal places, to the range equation

$$45000\sin(2\theta) = 1000.$$

We can also use the Double Angle Identities to solve equations with multiple angles.

Example 4.23 (Solving an Equation with a Double Angle Identity)
Consider the equation

$$\sin(2\theta) = \sin(\theta).$$

The fact that the two trigonometric functions have different periods makes this equation a little more difficult. We can use the Double Angle Identity for the sine to rewrite the equation as

$$2\sin(\theta)\cos(\theta) = \sin(\theta).$$

At this point we may be tempted to cancel the factor of $\sin(\theta)$ from both sides, but we should resist that temptation because $\sin(\theta)$ can be 0 and we can't divide by 0.

Instead, let's put everything on one side and factor:

$$\begin{aligned}2\sin(\theta)\cos(\theta) &= \sin(\theta)\\ 2\sin(\theta)\cos(\theta) - \sin(\theta) &= 0\\ \sin(\theta)(2\cos(\theta) - 1) &= 0.\end{aligned}$$

Now we have a product that is equal to 0, so at least one of the factors must be 0. This yields the two equations

$$\sin(\theta) = 0 \qquad\qquad 2\cos(\theta) - 1 = 0$$
$$\cos(\theta) = \frac{1}{2}$$

We solve each equation in turn. We know that $\sin(\theta) = 0$ when $\theta = k\pi$ for integers k. In the interval $[-\pi, \pi]$, the equation $\cos(\theta) = \frac{1}{2}$ has the two solutions $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3}$. So the solutions of the equation $\sin(2\theta) = \sin(\theta)$ are

$$\theta = k\pi \quad \text{or} \quad \theta = \frac{\pi}{3} + k(2\pi) \quad \text{or} \quad \theta = \frac{\pi}{3} + k(2\pi).$$

These solutions are illustrated in Figure 4.10.

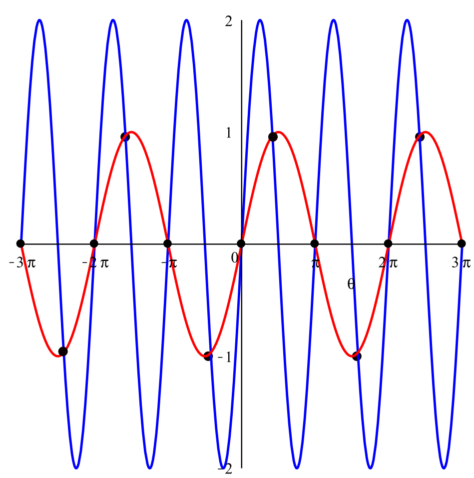


Figure 4.10: Graphs of $y = \sin(2\theta)$ and $y = \sin(\theta)$.

Progress Check 4.24 (Solving an Equation with a Double Angle Identity)
The goal is to solve the equation $\cos(2\theta) = \sin(\theta)$.

1. Use a double angle identity to help rewrite the equation in the form

$$2\sin^2(\theta) + \sin(\theta) - 1 = 0.$$

2. Solve the quadratic type equation in (1) by factoring the left side of the equation.

Half Angle Identities

Now we investigate the half angle identities, identities for $\cos\left(\dfrac{A}{2}\right)$ and $\sin\left(\dfrac{A}{2}\right)$. Here we use the double angle identities from Progress Check 4.20:

$$\begin{aligned}
\cos(A) &= \cos\left(2\left(\frac{A}{2}\right)\right) \\
\cos(A) &= 2\cos^2\left(\frac{A}{2}\right) - 1 \\
\cos(A) + 1 &= 2\cos^2\left(\frac{A}{2}\right) \\
\cos^2\left(\frac{A}{2}\right) &= \frac{\cos(A) + 1}{2} \\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 + \cos(A)}{2}}.
\end{aligned}$$

The sign of $\cos\left(\dfrac{A}{2}\right)$ depends on the quadrant in which $\dfrac{A}{2}$ lies.

Example 4.25 (Using the Cosine Half Angle Identity)
We can use the Cosine Half Angle Identity to determine the exact value of $\cos\left(\dfrac{7\pi}{12}\right)$. If we let $A = \dfrac{7\pi}{6}$, then we have $\dfrac{7\pi}{12} = \dfrac{A}{2}$. The Cosine Half Angle Identity shows

us that

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\frac{7\pi}{6}}{2}\right)\\ &= \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}\\ &= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \pm\sqrt{\frac{2-\sqrt{3}}{4}}.\end{aligned}$$

Since the terminal side of the angle $\frac{7\pi}{12}$ lies in the second quadrant, we know that $\cos\left(\frac{7\pi}{12}\right)$ is negative. Therefore,

$$\cos\left(\frac{7\pi}{12}\right) = -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}.$$

We can find a similar half angle formula for the sine using the same approach:

$$\begin{aligned}\cos(A) &= \cos\left(2\left(\frac{A}{2}\right)\right)\\ \cos(A) &= 1-2\sin^2\left(\frac{A}{2}\right)\\ \cos(A)-1 &= -2\sin^2\left(\frac{A}{2}\right)\\ \sin^2\left(\frac{A}{2}\right) &= \frac{1-\cos(A)}{2}\\ \sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1-\cos(A)}{2}}.\end{aligned}$$

Again, the sign of $\sin\left(\frac{A}{2}\right)$ depends on the quadrant in which $\frac{A}{2}$ lies.

To summarize,

Half Angle Identities
For any number A we have

- $\cos\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1+\cos(A)}{2}}$
- $\sin\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1-\cos(A)}{2}}$

where the sign depends on the quadrant in which $\dfrac{A}{2}$ lies.

Progress Check 4.26 (Using the Half Angle Identities)
Use a Half Angle Identity to find the exact value of $\cos\left(\dfrac{\pi}{8}\right)$.

Summary of Section 4.4

In this section, we studied the following important concepts and ideas:

- **Double Angle Identities**

$$\cos(2A) = \cos^2(A) - \sin^2(A) \qquad \sin(2A) = 2\cos(A)\sin(A)$$

$$\cos(2A) = 2\cos^2(A) - 1 \qquad \tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

- **Half Angle Identities**

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}} \qquad \sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

where the sign depends on the quadrant in which $\dfrac{A}{2}$ lies.

Exercises for Section 4.4

⋆ **1.** Given that $\cos(\theta) = \dfrac{2}{3}$ and $\sin(\theta) < 0$, determine the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

2. Find all solutions to the given equation. Use a graphing utility to graph each side of the equation to check your solutions.

⋆ **(a)** $\cos(x)\sin(x) = \dfrac{1}{2}$

(b) $\cos(2x) + 3 = 5\cos(x)$

3. Determine which of the following equations is an identity. Verify your responses.

⋆ **(a)** $\cot(t)\sin(2t) = 1 + \cos(2t)$

(b) $\sin(2x) = \dfrac{2 - \csc^2(x)}{\csc^2(x)}$

(c) $\cos(2x) = \dfrac{2 - \sec^2(x)}{\sec^2(x)}$

4. Find a simpler formula for $\cos(\pi + x)$ in terms of $\cos(x)$. Illustrate with a graph.

5. A classmate shares his solution to the problem of solving $\sin(2x) = 2\cos(x)$ over the interval $[0, 2\pi)$. He has written

$$\begin{aligned} \sin(2x) &= 2\cos(x) \\ \frac{\sin(2x)}{2} &= \cos(x) \\ \sin(x) &= \cos(x) \\ \tan(x) &= 1, \end{aligned}$$

so $x = \dfrac{\pi}{4}$ or $x = \dfrac{5\pi}{4}$.

(**a**) Draw graphs of $\sin(2x)$ and $2\cos(x)$ and explain why this classmates solution is incorrect.

(**b**) Find the error in this classmate's argument.

(**c**) Determine the solutions to $\sin(2x) = 2\cos(x)$ over the interval $[0, 2\pi)$.

6. Determine the exact value of each of the following:

⋆ **(a)** $\sin(22.5°)$

(b) $\cos(22.5°)$

⋆ **(c)** $\tan(22.5°)$

(d) $\sin(15°)$

(e) $\cos(15°)$

(f) $\tan(15°)$

(g) $\sin(195°)$

⋆ **(h)** $\cos(195°)$

(i) $\tan(195°)$

7. Determine the exact value of each of the following:

⋆ **(a)** $\sin\left(\frac{3\pi}{8}\right)$ **(d)** $\sin\left(\frac{5\pi}{8}\right)$ **(g)** $\sin\left(\frac{11\pi}{12}\right)$

(b) $\cos\left(\frac{3\pi}{8}\right)$ **(e)** $\cos\left(\frac{5\pi}{8}\right)$ ⋆ **(h)** $\cos\left(\frac{11\pi}{12}\right)$

⋆ **(c)** $\tan\left(\frac{3\pi}{8}\right)$ **(f)** $\tan\left(\frac{5\pi}{8}\right)$ **(i)** $\tan\left(\frac{11\pi}{12}\right)$

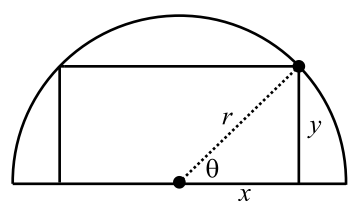
8. If $\cos(x) = \frac{2}{3}$ and $\sin(x) < 0$ and $0 \leq x \leq 2\pi$, determine the exact value of each of the following:

⋆ **(a)** $\cos\left(\frac{x}{2}\right)$ **(b)** $\sin\left(\frac{x}{2}\right)$ **(c)** $\tan\left(\frac{x}{2}\right)$

9. If $\sin(x) = \frac{2}{5}$ and $\cos(x) < 0$ and $0 \leq x \leq 2\pi$, determine the exact value of each of the following:

(a) $\cos\left(\frac{x}{2}\right)$ **(b)** $\sin\left(\frac{x}{2}\right)$ **(c)** $\tan\left(\frac{x}{2}\right)$

10. A rectangle is inscribed in a semicircle of radius r as shown in the diagram to the right.



We can write the area A of this rectangle as $A = (2x)y$.

(a) Write the area of this inscribed rectangle as a function of the angle θ shown in the diagram and then show that $A = r^2 \sin(2\theta)$.

(b) Use the formula from part (a) to determine the angle θ that produces the largest value of A and determine the dimensions of this inscribed rectangle with the largest possible area.

11. Derive the Triple Angle Identity

$$\sin(3A) = -4\sin^3(A) + 3\sin(A)$$

for the sine with the following steps.

(**a**) Write $3A$ as $2A + A$ and apply the Sine Sum Identity to write $\sin(3A)$ in terms of $\sin(2A)$ and $\sin(A)$.

(**b**) Use Double Angle Identities to write $\sin(2A)$ in terms of $\sin(A)$ and $\cos(A)$ and to write $\cos(2A)$ in terms of $\sin(A)$.

(**c**) Use a Pythagorean Identity to write $\cos^2(A)$ in terms of $\sin^2(A)$ and simplify.

12. Derive the Quadruple Angle Identity

$$\sin(4x) = 4\cos(x)\left[\sin(x) - 2\sin^3(x)\right]$$

as follows.

(**a**) Write $\sin(4x) = \sin(2(2x)$ and use the Double Angle Identity for sine to rewrite this formula.

(**b**) Now use the Double Angle Identity for sine and one of the Double Angle Identities for cosine to rewrite the expression from part (a).

(**c**) Algebraically rewrite the expression from part (b) to obtain the desired formula for $\sin(4x)$.

4.5 Sum and Product Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Product-to-Sum Identities for the sine and cosine?
- What are the Sum-to-Product Identities for the sine and cosine?
- Why are these identities useful?

In general, trigonometric equations are very difficult to solve exactly. We have been using identities to solve trigonometric equations, but there are still many more for which we cannot find exact solutions. Consider, for example, the equation

$$\sin(3x) + \sin(x) = 0.$$

The graph of $y = \sin(3x) + \sin(x)$ is shown in Figure 4.11. We can see that there are many solutions, but the identities we have so far do not help us with this equation. What would make this equation easier to solve is if we could rewrite the sum on the left as a product – then we could use the fact that a product is zero if and only if one of its factors is 0. We will later introduce the Sum-to-Product Identities that will help us solve this equation.

Beginning Activity

1. Let $A = 60°$ and $B = 30°$. Calculate

$$\cos(A)\cos(B) \quad \text{and} \quad \left(\frac{1}{2}\right)[\cos(A+B) + \cos(A-B)]$$

What do you notice?

Product-to-Sum Identities

The Cosine Sum and Difference Identities

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B) \quad (4)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B) \quad (5)$$

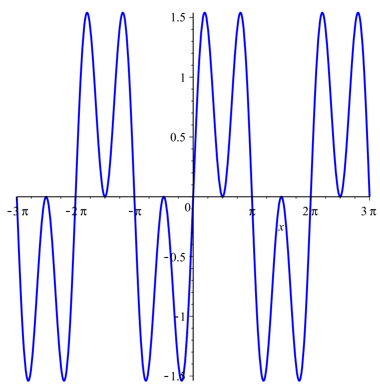


Figure 4.11: Graph of $y = \sin(3x) + \sin(x)$.

will allow us to develop identities that will express product of cosines or sines in terms of sums of cosines and sines. To see how these identities arise, we add the left and right sides of (4) and (5) to obtain

$$\cos(A - B) + \cos(A + B) = 2\cos(A)\cos(B).$$

So

$$\cos(A)\cos(B) = \left(\frac{1}{2}\right)[\cos(A + B) + \cos(A - B)].$$

Similarly, subtracting the left and right sides of (5) from (4) gives us

$$\cos(A - B) - \cos(A + B) = 2\sin(A)\sin(B).$$

So

$$\sin(A)\sin(B) = \left(\frac{1}{2}\right)[\cos(A - B) - \cos(A + B)].$$

We can similarly obtain a formula for $\cos(A)\sin(B)$. In this case, we use the sine sum and difference formulas

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B) \tag{6}$$
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B). \tag{7}$$

Adding the left and right hand sides of (6) and (7) yields

$$\sin(A - B) + \sin(A + B) = 2\sin(A)\cos(B).$$

So

$$\sin(A)\cos(B) = \left(\frac{1}{2}\right)\left[\sin(A+B) + \sin(A-B)\right].$$

Product-to-Sum Identities
For any numbers A and B we have

$$\cos(A)\cos(B) = \left(\frac{1}{2}\right)\left[\cos(A+B) + \cos(A-B)\right]$$
$$\sin(A)\sin(B) = \left(\frac{1}{2}\right)\left[\cos(A-B) - \cos(A+B)\right]$$
$$\sin(A)\cos(B) = \left(\frac{1}{2}\right)\left[\sin(A+B) + \sin(A-B)\right].$$

Progress Check 4.27 (Using the Product-to-Sum Identities)
Find the exact value of $\sin(52.5°)\sin(7.5°)$.

Sum-to-Product Identities

As our final identities, we derive the reverse of the Product-to-Sum identities. These identities are called the Sum-to-Product identities. For example, to verify the identity

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

we first note that $A = \dfrac{A+B}{2} + \dfrac{A-B}{2}$ and $B = \dfrac{A+B}{2} - \dfrac{A-B}{2}$. So

$$\begin{aligned}\cos(A) &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right)\\ &= \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \quad (8)\end{aligned}$$

and

$$\begin{aligned}\cos(B) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right)\\ &= \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right). \quad (9)\end{aligned}$$

Adding the left and right sides of (8) and (9) results in

$$\cos(A)+\cos(B)=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

Also, if we subtract the left and right hands sides of (9) from (8) we obtain

$$\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

Similarly,

$$\begin{aligned}\sin(A)&=\sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)\\&=\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\qquad(10)\end{aligned}$$

and

$$\begin{aligned}\sin(B)&=\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right)\\&=\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).\qquad(11)\end{aligned}$$

Adding the left and right sides of (10) and (11) results in

$$\sin(A)+\sin(B)=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

Again, if we subtract the left and right hands sides of (11) from (10) we obtain

$$\sin(A)-\sin(B)=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

Sum-to-Product Identities

For any numbers A and B we have

$$\begin{aligned}\cos(A)+\cos(B)&=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\\cos(A)-\cos(B)&=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\\\sin(A)+\sin(B)&=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\\sin(A)-\sin(B)&=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).\end{aligned}$$

Progress Check 4.28 (Using the Sum-to-Product Identities)
Find the exact value of $\cos(112.5°) + \cos(67.5°)$.

We can use these Sum-to-Product and Product-to-Sum Identities to solve even more types of trigonometric equations.

Example 4.29 (Solving Equations Using the Sum-to-Product Identity)
Let us return to the problem stated at the beginning of this section to solve the equation

$$\sin(3x) + \sin(x) = 0.$$

Using the Sum-to-Product Identity

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

with $A = 3x$ and $B = x$ we can rewrite the equation as follows:

$$\begin{aligned} \sin(3x) + \sin(x) &= 0 \\ 2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right) &= 0 \\ 2\sin(2x)\cos(x) &= 0. \end{aligned}$$

The advantage of this form is that we now have a product of functions equal to 0, and the only way a product can equal 0 is if one of the factors is 0. This reduces our original problem to two equations we can solve:

$$\sin(2x) = 0 \;\text{ or }\; \cos(x) = 0.$$

We know that $\sin(2x) = 0$ when $2x = k\pi$ or $x = k\frac{\pi}{2}$, where k is any integer. Also, $\cos\left(\frac{x}{2}\right) = 0$ when $\frac{x}{2} = \frac{\pi}{2} + k\pi$ or $x = \pi + k2\pi$, where k is any integer, but these are also solutions of the equation $\sin(2x) = 0$. So the solutions of $\sin(3x) + \sin(x) = 0$ are $x = k\frac{\pi}{2}$, where k is any integer. These solutions can be seen where the graph of $y = \sin(3x) + \sin(x)$ intersects the x-axis as illustrated in Figure 4.12.

Summary of Trigonometric Identities

Trigonometric identities are useful in that they allow us to determine exact values for the trigonometric functions at more points than before and also provide tools for deriving new identities and for solving trigonometric equations. Here we provide a summary of our trigonometric identities.

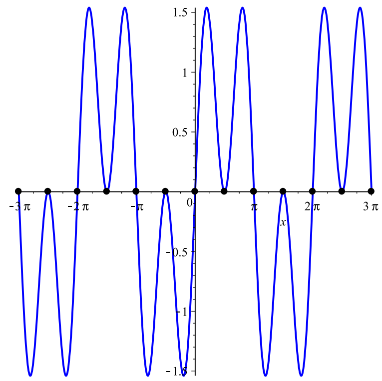


Figure 4.12: Graph of $y = \sin(3x) + \sin(x)$.

Cofunction Identities

$$\cos\left(\frac{\pi}{2} - A\right) = \sin(A)$$
$$\sin\left(\frac{\pi}{2} - A\right) = \cos(A)$$
$$\tan\left(\frac{\pi}{2} - A\right) = \cot(A).$$

Double Angle Identities

$$\sin(2A) = 2\cos(A)\sin(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}.$$

Half Angle Identities

$$\cos^2\left(\frac{A}{2}\right) = \frac{1 + \cos(A)}{2}$$
$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos(A)}{2}}$$
$$\sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos(A)}{2}$$
$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos(A)}{2}}$$
$$\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{1 + \cos(A)}$$
$$\tan\left(\frac{A}{2}\right) = \frac{1 - \cos(A)}{\sin(A)}.$$

The signs of $\cos\left(\frac{A}{2}\right)$ and $\sin\left(\frac{A}{2}\right)$ depend on the quadrant in which $\frac{A}{2}$ lies.

Cosine Difference and Sum Identities

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

Sine Difference and Sum Identities

$$\begin{aligned}
\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\\
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B).
\end{aligned}$$

Tangent Difference and Sum Identities

$$\begin{aligned}
\tan(A-B) &= \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}\\
\tan(A+B) &= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}.
\end{aligned}$$

Product-to-Sum Identities

$$\begin{aligned}
\cos(A)\cos(B) &= \left(\frac{1}{2}\right)[\cos(A+B)+\cos(A-B)]\\
\sin(A)\sin(B) &= \left(\frac{1}{2}\right)[\cos(A-B)-\cos(A-B)]\\
\sin(A)\cos(B) &= \left(\frac{1}{2}\right)[\sin(A+B)+\sin(A-B)].
\end{aligned}$$

Sum-to-Product Identities

$$\begin{aligned}
\cos(A)+\cos(B) &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\cos(A)-\cos(B) &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\\
\sin(A)+\sin(B) &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\sin(A)-\sin(B) &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).
\end{aligned}$$

Exercises for Section 4.5

1. Write each of the following expressions as a sum of trigonometric function values. When possible, determine the exact value of the resulting expression.

⋆ **(a)** $\sin(37.5°)\cos(7.5°)$

(b) $\sin(75°)\sin(15°)$

(c) $\cos(44°)\cos(16°)$

(d) $\cos(45°)\cos(15°)$

⋆ **(e)** $\cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)$

(f) $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$

2. Write each of the following expressions as a product of trigonometric function values. When possible, determine the exact value of the resulting expression.

⋆ **(a)** $\sin(50°) + \sin(10°)$

(b) $\sin(195°) - \sin(105°)$

(c) $\cos(195°) - \cos(15°)$

(d) $\cos(76°) + \cos(14°)$

⋆ **(e)** $\cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$

(f) $\sin\left(\frac{7\pi}{4}\right) - \sin\left(\frac{5\pi}{12}\right)$

3. Find all solutions to the given equation. Use a graphing utility to graph each side of the given equation to check your solutions.

⋆ **(a)** $\sin(2x) + \sin(x) = 0$

(b) $\sin(x)\cos(x) = \frac{1}{4}$

(c) $\cos(2x) + \cos(x) = 0$

Chapter 5

Complex Numbers and Polar Coordinates

One of the goals of algebra is to find solutions to polynomial equations. You have probably done this many times in the past, solving equations like $x^2 - 1 = 0$ or $2x^2 + 1 = 3x$. In the process, you encountered the quadratic formula that allows us to find all solutions to quadratic equations. For example, the quadratic formula gives us the solutions $x = \frac{2 + \sqrt{-4}}{2}$ and $x = \frac{2 - \sqrt{-4}}{2}$ for the quadratic equation $x^2 - 2x + 2 = 0$. In this chapter we will make sense of solutions like these that involve negative numbers under square roots, and discover connections between algebra and trigonometry that will allow us to solve a larger collection of polynomial equations than we have been able to until now.

5.1 The Complex Number System

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a complex number?
- What does it mean for two complex numbers to be equal?
- How do we add two complex numbers?
- How do we multiply two complex numbers?
- What is the conjugate of a complex number?
- What is the modulus of a complex number?
- How are the conjugate and modulus of a complex number related?
- How do we divide one complex number by another?

The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ allows us to find solutions of the quadratic equation $ax^2 + bx + c = 0$. For example, the solutions to the equation $x^2 + x + 1 = 0$ are

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

A problem arises immediately with this solution since there is no real number t with the property that $t^2 = -3$ or $t = \sqrt{-3}$. To make sense of solutions like this we introduce *complex numbers*. Although complex numbers arise naturally when solving quadratic equations, their introduction into mathematics came about from the problem of solving cubic equations.[1]

If we use the quadratic formula to solve an equation such as $x^2 + x + 1 = 0$,

[1] An interesting, and readable, telling of this history can be found in Chapter 6 of *Journey Through Genius* by William Dunham.

we obtain the solutions $x = \dfrac{-1+\sqrt{-3}}{2}$ and $x = \dfrac{-1-\sqrt{-3}}{2}$. These numbers are complex numbers and we have a special form for writing these numbers. We write them in a way that isolates the square root of -1. To illustrate, the number $\dfrac{-1+\sqrt{-3}}{2}$ can be written as follows;

$$\begin{aligned}\frac{-1+\sqrt{-3}}{2} &= -\frac{1}{2}+\frac{\sqrt{-3}}{2}\\ &= -\frac{1}{2}+\frac{\sqrt{3}\sqrt{-1}}{2}\\ &= -\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}.\end{aligned}$$

Since there is no real number t satisfying $t^2 = -1$, the number $\sqrt{-1}$ is not a real number. We call $\sqrt{-1}$ an *imaginary* number and give it the special label i. Thus, $i = \sqrt{-1}$ or $i^2 = -1$. With this in mind we can write

$$\frac{-1+\sqrt{-3}}{2} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i,$$

and every complex number has this special form.

Definition. A **complex number** is an object of the form

$$a + bi,$$

where a and b are real numbers and $i^2 = -1$.

The form $a + bi$, where a and b are real numbers is called the **standard form** for a complex number. When we have a complex number of the form $z = a + bi$, the number a is called the **real part** of the complex number z and the number b is called the **imaginary part** of z. Since i is not a real number, two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

There is an arithmetic of complex numbers that is determined by an addition and multiplication of complex numbers. Adding and subtracting complex numbers is natural:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

That is, to add (or subtract) two complex numbers we add (subtract)their real parts and add (subtract) their imaginary parts. Multiplication is also done in a natural way – to multiply two complex numbers, we simply expand the product as usual and exploit the fact that $i^2 = -1$. So the product of two complex number is

$$\begin{aligned}(a + bi)(c + di) &= ac + a(di) + (bi)c + (bi)(di) \\ &= ac + (ad)i + (bc)i + (bd)i^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

It can be shown that the complex numbers satisfy many useful and familiar properties, which are similar to properties of the real numbers. If u, w, and z, are complex numbers, then

1. $w + z = z + w$.

2. $u + (w + z) = (u + w) + z$.

3. The complex number $0 = 0 + 0i$ is an additive identity, that is $z + 0 = z$.

4. If $z = a + bi$, then the additive inverse of z is $-z = (-a) + (-b)i$, That is, $z + (-z) = 0$.

5. $wz = zw$.

6. $u(wz) = (uw)z$.

7. $u(w + z) = uw + uz$.

8. If $wz = 0$, then $w = 0$ or $z = 0$.

We will use these properties as needed. For example, to write the complex product $(1 + i)i$ in the form $a + bi$ with a and b real numbers, we distribute multiplication over addition and use the fact that $i^2 = -1$ to see that

$$(1 + i)i = i + i^2 = i + (-1) = (-1) + i.$$

For another example, if $w = 2 + i$ and $z = 3 - 2i$, we can use these properties to write wz in the standard $a + bi$ form as follows:

$$\begin{aligned} wz &= (2+i)z \\ &= 2z + iz \\ &= 2(3-2i) + i(3-2i) \\ &= (6-4i) + (3i - 2i^2) \\ &= 6 - 4i + 3i - 2(-1) \\ &= 8 - i \end{aligned}$$

Progress Check 5.1 (Sums and Products of Complex Numbers)

1. Write each of the sums or products as a complex number in standard form.

   (**a**) $(2+3i) + (7-4i)$

   (**b**) $(4-2i)(3+i)$

   (**c**) $(2+i)i - (3+4i)$

2. Use the quadratic formula to write the two solutions to the quadratic equation $x^2 - x + 2 = 0$ as complex numbers of the form $r + si$ and $u + vi$ for some real numbers r, s, u, and v. (**Hint**: Remember: $i = \sqrt{-1}$. So we can rewrite something like $\sqrt{-4}$ as $\sqrt{-4} = \sqrt{4}\,\sqrt{-1} = 2i$.)

Geometric Representations of Complex Numbers

Each ordered pair (a, b) of real numbers determines:

- A point in the coordinate plane with coordinates (a, b).
- A complex number $a + bi$.
- A vector $a\mathbf{i} + b\mathbf{j} = \langle a, b \rangle$.

This means that we can geometrically represent the complex number $a + bi$ with a vector in standard position with terminal point (a, b). Therefore, we can draw pictures of complex numbers in the plane. When we do this, the horizontal axis is called **the real axis**, and the vertical axis is called **the imaginary axis**. In addition, the coordinate plane is then referred to as **the complex plane**. That is, if $z = a+bi$ we can think of z as a directed line segment from the origin to the point (a, b),

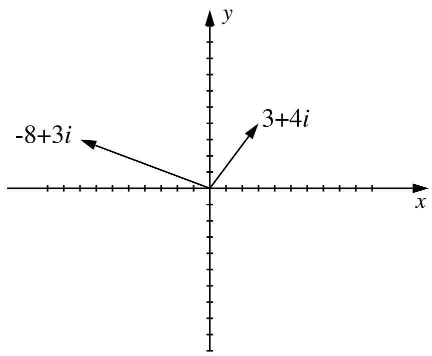


Figure 5.1: Two complex numbers.

where the terminal point of the segment is a units from the imaginary axis and b units from the real axis. For example, the complex numbers $3 + 4i$ and $-8 + 3i$ are shown in Figure 5.1.

In addition, the sum of two complex numbers can be represented geometrically using the vector forms of the complex numbers. Draw the parallelogram defined by $w = a+bi$ and $z = c+di$. The sum of w and z is the complex number represented by the vector from the origin to the vertex on the parallelogram opposite the origin as illustrated with the vectors $w = 3 + 4i$ and $z = -8 + 3i$ in Figure 5.2.

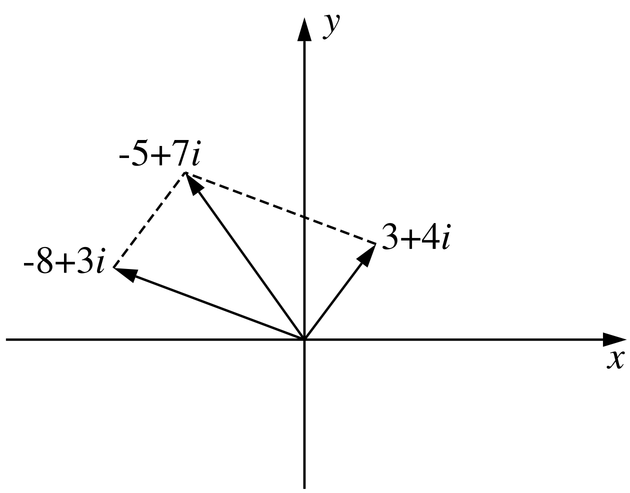


Figure 5.2: The Sum of Two Complex Numbers.

Progress Check 5.2 (Visualizing Complex Addition)
Let $w = 2 + 3i$ and $z = -1 + 5i$.

1. Write the complex sum $w + z$ in standard form.
2. Draw a picture to illustrate the sum using vectors to represent w and z.

We now extend our use of the representation of a complex number as a vector in standard position to include the notion of the length of a vector. Recall from Section 3.6 (page 234) that the length of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. When we use this idea with complex numbers, we call it the *norm* or *modulus* of the complex number.

Definition. The **norm** (or **modulus**) of the complex number $z = a + bi$ is the distance from the origin to the point (a, b) and is denoted by $|z|$. We see that

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

There is another concept related to complex number that is based on the following bit of algebra.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2 \\ &= a^2 - b^2 i^2 \\ &= a^2 + b^2\end{aligned}$$

The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. If we let $z = a + bi$, we denote the complex conjugate of z as $\overline{z}$. So

$$\overline{z} = \overline{a + bi} = a - bi.$$

We also notice that

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2,$$

and so the product of a complex number with its conjugate is a real number. In fact,

$$\begin{aligned}&z\overline{z} = a^2 + b^2 = |z|^2, \text{ and so} \\ &|z| = \sqrt{z\overline{z}}\end{aligned}$$

Progress Check 5.3 (Operations on Complex Numbers)
Let $w = 2 + 3i$ and $z = -1 + 5i$.

1. Find $\overline{w}$ and $\overline{z}$.
2. Compute $|w|$ and $|z|$.
3. Compute $w\overline{w}$ and $z\overline{z}$.
4. What is $\overline{z}$ if z is a real number?

Division of Complex Numbers

We can add, subtract, and multiply complex numbers, so it is natural to ask if we can divide complex numbers. We illustrate with an example.

Example 5.4 (Dividing by a Complex Number)
Suppose we want to write the quotient $\dfrac{2+i}{3+i}$ as a complex number in the form $a + bi$. This problem is rationalizing a denominator since $i = \sqrt{-1}$. So in this case we need to "remove" the imaginary part from the denominator. Recall that the product of a complex number with its conjugate is a real number, so if we multiply the numerator and denominator of $\dfrac{2+i}{3+i}$ by the complex conjugate of the denominator, we can rewrite the denominator as a real number. The steps are as follows. Multiplying the numerator and denominator by the conjugate of $3 + i$, which is $3 - i$. This gives us

$$\begin{aligned}
\frac{2+i}{3+i} &= \left(\frac{2+i}{3+i}\right)\left(\frac{3-i}{3-i}\right) \\
&= \frac{(2+i)(3-i)}{(3+i)(3-i)} \\
&= \frac{(6-i^2)+(-2+3)i}{9-i^2} \\
&= \frac{7+i}{10}.
\end{aligned}$$

Now we can write the final result in standard form as $\dfrac{7+i}{10} = \dfrac{7}{10} + \dfrac{1}{10}i$.

Example 5.4 illustrates the general process for dividing one complex number by another. In general, we can write the quotient $\dfrac{a+bi}{c+di}$ in the form $r + si$ by multiplying numerator and denominator of our fraction by the conjugate $c - di$ of $c + di$ to see that

$$\begin{aligned}\frac{a+bi}{c+di} &= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.\end{aligned}$$

Therefore, we have the formula for the quotient of two complex numbers.

The **quotient** $\dfrac{a+bi}{c+di}$ of the complex numbers $a + bi$ and $c + di$ is the complex number

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i,$$

provided $c + di \neq 0$.

Progress Check 5.5 (Dividing Complex Numbers)
Let $z = 3 + 4i$ and $w = 5 - i$.

1. Write $\dfrac{w}{z} = \dfrac{5-i}{3+4i}$ as a complex number in the form $r + si$ where r and s are some real numbers. Check the result by multiplying the quotient by $3 + 4i$. Is this product equal to $5 - i$?

2. Find the solution to the equation $(3 + 4i)x = 5 - i$ as a complex number in the form $x = u + vi$ where u and v are some real numbers.

Summary of Section 5.1

In this section, we studied the following important concepts and ideas:

- A **complex number** is an object of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. When we have a complex number of the form $z = a + bi$, the number a is called the **real part** of the complex number z and the number b is called the **imaginary part** of z.

- We can add, subtract, multiply, and divide complex numbers as follows:

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\(a+bi)+(c+di) &= (a+c)+(b+d)i\\(a+bi)(c+di) &= (ac-bd)+(ad+bc)i\\\frac{a+bi}{c+di} &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i, \text{ provided } c+di \neq 0\end{aligned}$$

- A complex number $a+bi$ can be represented geometrically with a vector in standard position with terminal point (a, b). When we do this, the horizontal axis is called **the real axis**, and the vertical axis is called **the imaginary axis**. In addition, the coordinate plane is then referred to as **the complex plane**. That is, if $z = a+bi$ we can think of z as a directed line segment from the origin to the point (a, b), where the terminal point of the segment is a units from the imaginary axis and b units from the real axis.

- The **norm** (or **modulus**) of the complex number $z = a+bi$ is the distance from the origin to the point (a, b) and is denoted by $|z|$. We see that

$$|z| = |a+bi| = \sqrt{a^2+b^2}.$$

- The complex number $a-bi$ is called the **complex conjugate** of $a+bi$. Note that

$$(a+bi)(a-bi) = a^2+b^2 = |a+bi|^2.$$

Exercises for Section 5.1

* **1.** Write each of the following as a complex number in standard form.

(**a**) $(4+i)+(3-3i)$	(**c**) $(4+2i)(5-3i)$
(**b**) $5(2-i)+i(3-2i)$	(**d**) $(2+3i)(1+i)+(4-3i)$

2. Use the quadratic formula to write the two solutions of each of the following quadratic equations in standard form.

⋆ **(a)** $x^2 - 3x + 5 = 0$

(b) $2x^2 = x - 7$

3. For each of the following pairs of complex numbers w and z, determine the sum $w + z$ and illustrate the sum with a diagram.

⋆ **(a)** $w = 3 + 2i, z = 5 - 4i$.

⋆ **(b)** $w = 4i, z = -3 + 2i$.

(c) $w = 5, z = -7 + 2i$.

(d) $w = 6 - 3i, z = -1 + 7i$.

4. For each of the following complex numbers z, determine $\overline{z}$, $|z|$, and $z\overline{z}$.

⋆ **(a)** $z = 5 + 2i$

⋆ **(b)** $z = 3i$

(c) $z = 3 - 4i$

(d) $z = 7 + i$

5. Write each of the following quotients in standard form.

⋆ **(a)** $\dfrac{5 + i}{3 + 2i}$

⋆ **(b)** $\dfrac{3 + 3i}{i}$

(c) $\dfrac{i}{2 - i}$

(d) $\dfrac{4 + 2i}{1 - i}$

6. We know that $i^1 = i$ and $i^2 = -1$. We can then see that

$$i^3 = i^2 \cdot i = (-1)i = -i.$$

(a) Show that $i^4 = 1$.

(b) Now determine i^5, i^6, i^7, and i^8. **Note**: Each power of i will equal 1, -1, i, or $-i$.

(c) Notice that $13 = 4 \cdot 3 + 1$. We will use this to determine i^{13}.

$$i^{13} = i^{4 \cdot 3 + 1} = i^{4 \cdot 3} i^1 = \left(i^4\right)^3 \cdot i$$

So what is i^{13}?

(d) Using $39 = 4 \cdot 9 + 3$, determine i^{39}.

(e) Determine i^{54}.

7. **(a)** Write the complex number $i(2+2i)$ in standard form. Plot the complex numbers $2 + 2i$ and $i(2 + 2i)$ in the complex plane. What appears to be the angle between these two complex numbers?

(b) Repeat part (a) for the complex numbers $2 - 3i$ and $i(2 - 3i)$.

(**c**) Repeat part (a) for the complex numbers $3i$ and $i(3i)$.

(**d**) Describe what happens when the complex number $a + bi$ is multiplied by the complex number i.

5.2 The Trigonometric Form of a Complex Number

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the polar (trigonometric) form of a complex number?
- How do we multiply two complex numbers in polar form?
- How do we divide one complex number in polar form by a nonzero complex number in polar form?

Multiplication of complex numbers is more complicated than addition of complex numbers. To better understand the product of complex numbers, we first investigate the trigonometric (or polar) form of a complex number. This trigonometric form connects algebra to trigonometry and will be useful for quickly and easily finding powers and roots of complex numbers.

Beginning Activity

If $z = a + bi$ is a complex number, then we can plot z in the plane as shown in Figure 5.3. In this situation, we will let r be the magnitude of z (that is, the distance from z to the origin) and θ the angle z makes with the positive real axis as shown in Figure 5.3.

1. Use right triangle trigonometry to write a and b in terms of r and θ.

2. Explain why we can write z as

$$z = r(\cos(\theta) + i\,\sin(\theta)). \tag{1}$$

When we write z in the form given in Equation (1), we say that z is written in **trigonometric form** (or *polar form*).[2] The angle θ is called the **argument** of the

[2] The word *polar* here comes from the fact that this process can be viewed as occurring with polar coordinates.

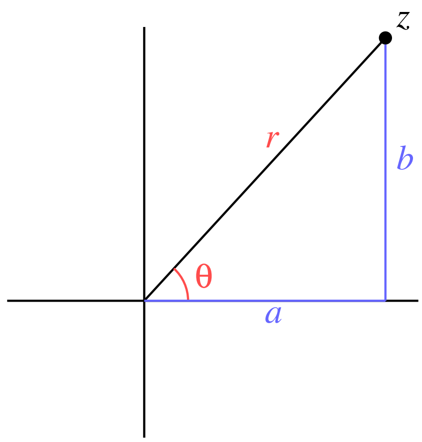


Figure 5.3: Trigonometric form of a complex number.

complex number z and the real number r is the **modulus** or **norm** of z. To find the polar representation of a complex number $z = a + bi$, we first notice that

$$r = |z| = \sqrt{a^2 + b^2}$$
$$a = r\cos(\theta)$$
$$b = r\sin(\theta)$$

To find θ, we have to consider cases.

- If $z = 0 = 0 + 0i$, then $r = 0$ and θ can have any real value.
- If $z \neq 0$ and $a \neq 0$, then $\tan(\theta) = \dfrac{b}{a}$.
- If $z \neq 0$ and $a = 0$ (so $b \neq 0$), then
  - * $\theta = \dfrac{\pi}{2}$ if $b > 0$
  - * $\theta = -\dfrac{\pi}{2}$ if $b < 0$.

Progress Check 5.6 (The Polar Form of a Complex Number)

1. Determine the polar form of the complex numbers $w = 4 + 4\sqrt{3}i$ and $z = 1 - i$.

2. Determine real numbers a and b so that $a + bi = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$.

There is an alternate representation that you will often see for the polar form of a complex number using a complex exponential. We won't go into the details, but only consider this as notation. When we write $e^{i\theta}$ (where i is the complex number with $i^2 = -1$) we mean

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

So the polar form $r(\cos(\theta) + i\sin(\theta))$ can also be written as $re^{i\theta}$:

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta)).$$

Products of Complex Numbers in Polar Form

There is an important product formula for complex numbers that the polar form provides. We illustrate with an example.

Example 5.7 (Products of Complex Numbers in Polar Form)
Let $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = \sqrt{3} + i$. Using our definition of the product of complex numbers we see that

$$\begin{aligned} wz &= \left(\sqrt{3} + i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\sqrt{3} + i. \end{aligned}$$

Now we write w and z in polar form. Note that $|w| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and the argument of w satisfies $\tan(\theta) = -\sqrt{3}$. Since w is in the second quadrant, we see that $\theta = \frac{2\pi}{3}$, so the polar form of w is

$$w = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right).$$

Also, $|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ and the argument of z satisfies $\tan(\theta) = \frac{1}{\sqrt{3}}$. Since z is in the first quadrant, we know that $\theta = \frac{\pi}{6}$ and the polar form of z is

$$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right].$$

We can also find the polar form of the complex product wz. Here we have $|wz| = 2$, and the argument of wz satisfies $\tan(\theta) = -\dfrac{1}{\sqrt{3}}$. Since wz is in quadrant II, we see that $\theta = \dfrac{5\pi}{6}$ and the polar form of wz is

$$wz = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right].$$

When we compare the polar forms of w, z, and wz we might notice that $|wz| = |w|\,|z|$ and that the argument of zw is $\dfrac{2\pi}{3} + \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ or the sum of the arguments of w and z. This turns out to be true in general.

The result of Example 5.7 is no coincidence, as we will show. In general, we have the following important result about the product of two complex numbers.

Multiplication of Complex Numbers in Polar Form
Let $w = r(\cos(\alpha) + i\,\sin(\alpha))$ and $z = s(\cos(\beta) + i\,\sin(\beta))$ be complex numbers in polar form. Then the polar form of the complex product wz is given by

$$wz = rs\left(\cos(\alpha + \beta) + i\,\sin(\alpha + \beta)\right).$$

This states that to multiply two complex numbers in polar form, we multiply their norms and add their arguments.

To understand why this result it true in general, let $w = r(\cos(\alpha) + i\,\sin(\alpha))$ and $z = s(\cos(\beta) + i\,\sin(\beta))$ be complex numbers in polar form. We will use cosine and sine of sums of angles identities to find wz:

$$\begin{aligned} wz &= [r(\cos(\alpha) + i\,\sin(\alpha))][s(\cos(\beta) + i\,\sin(\beta))] \\ &= rs([\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i\,[\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)] \end{aligned} \tag{1}$$

We now use the cosine and sum identities (see page 293) and see that

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \text{ and}$$
$$\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)$$

Using equation (1) and these identities, we see that

$$\begin{aligned} wz &= rs([\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i\,[\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)] \\ &= rs(\cos(\alpha + \beta) + i\,\sin(\alpha + \beta)) \end{aligned}$$

as expected.

An illustration of this is given in Figure 5.4. The formula for multiplying complex numbers in polar form tells us that to multiply two complex numbers, we add their arguments and multiply their norms.

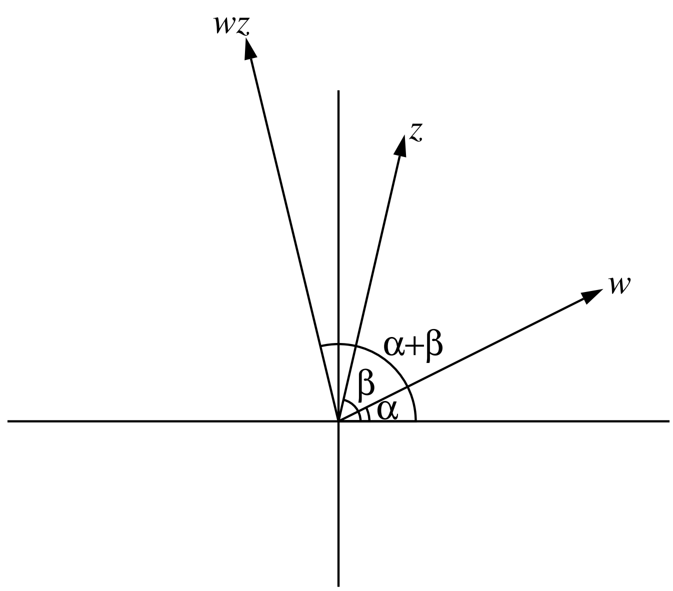


Figure 5.4: A Geometric Interpretation of Multiplication of Complex Numbers.

Progress Check 5.8 (Visualizing the Product of Complex Numbers)
Let $w = 3\left[\cos\left(\dfrac{5\pi}{3}\right) + i\sin\left(\dfrac{5\pi}{3}\right)\right]$ and $z = 2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$.

1. What is $|wz|$?
2. What is the argument of wz?
3. In which quadrant is wz? Explain.
4. Determine the polar form of wz.
5. Draw a picture of w, z, and wz that illustrates the action of the complex product.

Quotients of Complex Numbers in Polar Form

We have seen that we multiply complex numbers in polar form by multiplying their norms and adding their arguments. There is a similar method to divide one complex number in polar form by another complex number in polar form.

> **Division of Complex Numbers in Polar Form**
> Let $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ be complex numbers in polar form with $z \neq 0$. Then the polar form of the complex quotient $\dfrac{w}{z}$ is given by
>
> $$\frac{w}{z} = \frac{r}{s}\left(\cos(\alpha - \beta) + i\sin(\alpha - \beta)\right).$$
>
> So to divide complex numbers in polar form, we divide the norm of the complex number in the numerator by the norm of the complex number in the denominator and subtract the argument of the complex number in the denominator from the argument of the complex number in the numerator.

The proof of this is similar to the proof for multiplying complex numbers and is included as a supplement to this section.

Progress Check 5.9 (Visualizing the Quotient of Two Complex Numbers)
Let $w = 3\left[\cos\left(\dfrac{5\pi}{3}\right) + i\sin\left(\dfrac{5\pi}{3}\right)\right]$ and $z = 2\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$.

1. What is $\left|\dfrac{w}{z}\right|$?
2. What is the argument of $\left|\dfrac{w}{z}\right|$?
3. In which quadrant is $\left|\dfrac{w}{z}\right|$? Explain.
4. Determine the polar form of $\left|\dfrac{w}{z}\right|$.
5. Draw a picture of w, z, and $\left|\dfrac{w}{z}\right|$ that illustrates the action of the complex product.

Proof of the Rule for Dividing Complex Numbers in Polar Form

Let $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ be complex numbers in polar form with $z \neq 0$. So

$$\frac{w}{z} = \frac{r(\cos(\alpha) + i\sin(\alpha))}{s(\cos(\beta) + i\sin(\beta))} = \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}\right].$$

We will work with the fraction $\dfrac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}$ and follow the usual practice of multiplying the numerator and denominator by $\cos(\beta) - i\sin(\beta)$. So

$$\begin{aligned}
\frac{w}{z} &= \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}\right] \\
&= \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)} \cdot \frac{\cos(\beta) - i\sin(\beta)}{\cos(\beta) - i\sin(\beta)}\right] \\
&= \frac{r}{s}\left[\frac{(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta))}{\cos^2(\beta) + \sin^2(\beta)}\right]
\end{aligned}$$

We now use the following identities with the last equation:

- $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$.
- $\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \sin(\alpha - \beta)$.
- $\cos^2(\beta) + \sin^2(\beta) = 1$.

Using these identities with the last equation for $\dfrac{w}{z}$, we see that

$$\frac{w}{z} = \frac{r}{s}\left[\frac{\cos(\alpha - \beta) + i\sin(\alpha - \beta)}{1}\right] = \frac{r}{s}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)].$$

Summary of Section 5.2

In this section, we studied the following important concepts and ideas:

- If $z = a + bi$ is a complex number, then we can plot z in the plane. If r is the magnitude of z (that is, the distance from z to the origin) and θ the angle z makes with the positive real axis, then the **trigonometric form** (or **polar form**) of z is $z = r(\cos(\theta) + i\sin(\theta))$, where

  $$r = \sqrt{a^2 = b^2}, \cos(\theta) = \frac{a}{r}, \text{ and } \sin(\theta) = \frac{b}{r}.$$

  The angle θ is called the **argument** of the complex number z and the real number r is the **modulus** or **norm** of z.

- If $w = r(\cos(\alpha) + i\,\sin(\alpha))$ and $z = s(\cos(\beta) + i\,\sin(\beta))$ are complex numbers in polar form, then the polar form of the complex product wz is given by
$$wz = rs\,(\cos(\alpha+\beta) + i\,\sin(\alpha+\beta))\,,$$
and if $z \neq 0$, the polar form of the complex quotient $\dfrac{w}{z}$ is
$$\frac{w}{z} = \frac{r}{s}\,(\cos(\alpha-\beta) + i\,\sin(\alpha-\beta))\,,$$
This states that to multiply two complex numbers in polar form, we multiply their norms and add their arguments, and to divide two complex numbers, we divide their norms and subtract their arguments.

Exercises for Section 5.2

1. Determine the polar (trigonometric) form of each of the following complex numbers.

$\star$ **(a)** $3+3i$

(b) $3-3i$

(c) $-3+3i$

(d) $5i$

$\star$ **(e)** $4\sqrt{3}+4i$

(f) $-4\sqrt{3}-4i$

2. In each of the following, a complex number z is given. In each case, determine real numbers a and b so that $z = a + bi$. If it is not possible to determine exact values for a and b, determine the values of a and b correct to four decimal places.

$\star$ **(a)** $z = 5\left(\cos\left(\dfrac{\pi}{2}\right) + i\,\sin\left(\dfrac{\pi}{2}\right)\right)$

$\star$ **(b)** $z = 2.5\left(\cos\left(\dfrac{\pi}{4}\right) + i\,\sin\left(\dfrac{\pi}{4}\right)\right)$

(c) $z = 2.5\left(\cos\left(\dfrac{3\pi}{4}\right) + i\,\sin\left(\dfrac{3\pi}{4}\right)\right)$

(d) $z = 3\left(\cos\left(\dfrac{7\pi}{6}\right) + i\,\sin\left(\dfrac{7\pi}{6}\right)\right)$

(e) $z = 8\left(\cos\left(\dfrac{7\pi}{10}\right) + i\,\sin\left(\dfrac{7\pi}{10}\right)\right)$

3. For each of the following, write the product wz in polar (trigonometric form). When it is possible, write the product in form $a + bi$, where a and b are real numbers and do not involve a trigonometric function.

⋆ **(a)** $w = 5\left(\cos\left(\frac{\pi}{12}\right) + i\,\sin\left(\frac{\pi}{12}\right)\right)$, $z = 2\left(\cos\left(\frac{5\pi}{12}\right) + i\,\sin\left(\frac{5\pi}{12}\right)\right)$

⋆ **(b)** $w = 2.3\left(\cos\left(\frac{\pi}{3}\right) + i\,\sin\left(\frac{\pi}{3}\right)\right)$, $z = 3\left(\cos\left(\frac{5\pi}{4}\right) + i\,\sin\left(\frac{5\pi}{4}\right)\right)$

(c) $w = 2\left(\cos\left(\frac{7\pi}{10}\right) + i\,\sin\left(\frac{7\pi}{10}\right)\right)$, $z = 2\left(\cos\left(\frac{2\pi}{5}\right) + i\,\sin\left(\frac{2\pi}{5}\right)\right)$

(d) $w = (\cos(24°) + i\,\sin(24°))$, $z = 2(\cos(33°) + i\,\sin(33°))$

(e) $w = 2(\cos(72°) + i\,\sin(72°))$, $z = 2(\cos(78°) + i\,\sin(78°))$

⋆ **4.** For the complex numbers in Exercise (3), write the quotient $\frac{w}{z}$ in polar (trigonometric form). When it is possible, write the quotient in form $a + bi$, where a and b are real numbers and do not involve a trigonometric function.

5. (**a**) Write the complex number i in polar form.

(**b**) Let $z = r(\cos(\theta) + i\,\sin(\theta))$. Determine the product $i \cdot z$ in polar form. Use this to explain why the complex number $i \cdot z$ and z will be perpendicular when both are plotted in the complex plane.

5.3 DeMoivre's Theorem and Powers of Complex Numbers

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is de Moivre's Theorem and why is it useful?
- If n is a positive integer, what is an nth root of a complex number? How many nth roots does a complex number have? How do we find all of the nth roots of a complex number?

The trigonometric form of a complex number provides a relatively quick and easy way to compute products of complex numbers. As a consequence, we will be able to quickly calculate powers of complex numbers, and even roots of complex numbers.

Beginning Activity

Let $z = r(\cos(\theta) + i\sin(\theta))$. Use the trigonometric form of z to show that

$$z^2 = r^2\left(\cos(2\theta) + i\sin(2\theta)\right). \tag{1}$$

De Moivre's Theorem

The result of Equation (1) is not restricted to only squares of a complex number. If $z = r(\cos(\theta) + i\sin(\theta))$, then it is also true that

$$\begin{aligned} z^3 &= zz^2 \\ &= (r)(r^2)\left(\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)\right) \\ &= r^3\left(\cos(3\theta) + i\sin(3\theta)\right). \end{aligned}$$

We can continue this pattern to see that

$$\begin{aligned} z^4 &= zz^3 \\ &= (r)(r^3)\left(\cos(\theta + 3\theta) + i\sin(\theta + 3\theta)\right) \\ &= r^4\left(\cos(4\theta) + i\sin(4\theta)\right). \end{aligned}$$

The equations for z^2, z^3, and z^4 establish a pattern that is true in general. The result is called de Moivre's Theorem.

DeMoivre's Theorem
Let $z = r(\cos(\theta) + i\,\sin(\theta))$ be a complex number and n any integer. Then

$$z^n = r^n(\cos(n\theta) + i\,\sin(n\theta)).$$

It turns out that DeMoivre's Theorem also works for negative integer powers as well.

Progress Check 5.10 (DeMoivre's Theorem)
Write the complex number $1 - i$ in polar form. Then use DeMoivre's Theorem to write $(1 - i)^{10}$ in the complex form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

Roots of Complex Numbers

DeMoivre's Theorem is very useful in calculating powers of complex numbers, even fractional powers. We illustrate with an example.

Example 5.11 (Roots of Complex Numbers)
We will find all of the solutions to the equation $x^3 - 1 = 0$. These solutions are also called the *roots* of the polynomial $x^3 - 1$. To solve the equation $x^3 - 1 = 0$, we add 1 to both sides to rewrite the equation in the form $x^3 = 1$. Recall that to solve a polynomial equation like $x^3 = 1$ means to find all of the numbers (real or complex) that satisfy the equation. We can take the real cube root of both sides of this equation to obtain the solution $x_0 = 1$, but every cubic polynomial should have three solutions. How can we find the other two? If we draw the graph of $y = x^3 - 1$ we see that the graph intersects the x-axis at only one point, so there is only one real solution to $x^3 = 1$. That means the other two solutions must be complex and we can use DeMoivre's Theorem to find them. To do this, suppose

$$z = r\,[\cos(\theta) + i\,\sin(\theta)]$$

is a solution to $x^3 = 1$. Then

$$1 = z^3 = r^3(\cos(3\theta) + i\,\sin(3\theta)).$$

This implies that $r = 1$ (or $r = -1$, but we can incorporate the latter case into our choice of angle). We then reduce the equation $x^3 = 1$ to the equation

$$1 = \cos(3\theta) + i\,\sin(3\theta). \qquad (2)$$

Equation (2) has solutions when $\cos(3\theta) = 1$ and $\sin(3\theta) = 0$. This will occur when $3\theta = 2\pi k$, or $\theta = \frac{2\pi k}{3}$, where k is any integer. The distinct values of $\frac{2\pi k}{3}$ on the unit circle occur when $k = 0$ and $\theta = 0$, $k = 1$ and $\theta = \frac{2\pi}{3}$, and $k = 2$ with $\theta = \frac{4\pi}{3}$. In other words, the solutions to $x^3 = 1$ should be

$$\begin{aligned}
x_0 &= \cos(0) + i\ \sin(0) = 1 \\
x_1 &= \cos\left(\frac{2\pi}{3}\right) + i\ \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
x_2 &= \cos\left(\frac{4\pi}{3}\right) + i\ \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

We already know that $x_0^3 = 1^3 = 1$, so x_0 actually is a solution to $x^3 = 1$. To check that x_1 and x_2 are also solutions to $x^3 = 1$, we apply DeMoivre's Theorem:

$$\begin{aligned}
x_1^3 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\ \sin\left(\frac{2\pi}{3}\right)\right]^3 \\
&= \cos\left(3\left(\frac{2\pi}{3}\right)\right) + i\ \sin\left(3\left(\frac{2\pi}{3}\right)\right) \\
&= \cos(2\pi) + i\ \sin(2\pi) \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
x_2^3 &= \left[\cos\left(\frac{4\pi}{3}\right) + i\ \sin\left(\frac{4\pi}{3}\right)\right]^3 \\
&= \cos\left(3\left(\frac{4\pi}{3}\right)\right) + i\ \sin\left(3\left(\frac{4\pi}{3}\right)\right) \\
&= \cos(4\pi) + i\ \sin(4\pi) \\
&= 1.
\end{aligned}$$

Thus, $x_1^3 = 1$ and $x_2^3 = 1$ and we have found three solutions to the equation $x^3 = 1$. Since a cubic can have only three solutions, we have found them all.

The general process of solving an equation of the form $x^n = a + bi$, where n is a positive integer and $a + bi$ is a complex number works the same way. Write $a + bi$ in trigonometric form

$$a + bi = r\left[\cos(\theta) + i\ \sin(\theta)\right],$$

and suppose that $z = s[\cos(\alpha) + i\sin(\alpha)]$ is a solution to $x^n = a + bi$. Then

$$\begin{aligned} a + bi &= z^n \\ r[\cos(\theta) + i\sin(\theta)] &= (s[\cos(\alpha) + i\sin(\alpha)])^n \\ r[\cos(\theta) + i\sin(\theta)] &= s^n[\cos(n\alpha) + i\sin(n\alpha)] \end{aligned}$$

Using the last equation, we see that

$$s^n = r \quad \text{and} \quad \cos(\theta) + i\sin(\theta) = \cos(n\alpha) + i\sin(n\alpha).$$

Therefore,

$$s^n = r \quad \text{and} \quad n\alpha = \theta + 2\pi k$$

where k is any integer. This give us

$$s = \sqrt[n]{r} \quad \text{and} \quad \alpha = \frac{\theta + 2\pi k}{n}.$$

We will get n different solutions for $k = 0, 1, 2, \ldots, n-1$, and these will be all of the solutions. These solutions are called the ***n*th roots of the complex number *a + bi***. We summarize the results.

Roots of Complex Numbers
Let n be a positive integer. The nth roots of the complex number $r[\cos(\theta) + i\sin(\theta)]$ are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, (n-1)$.

If we want to represent the nth roots of $r[\cos(\theta) + i\sin(\theta)]$ using degrees instead of radians, the roots will have the form

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, (n-1)$.

Example 5.12 (Square Roots of 1)
As another example, we find the complex square roots of 1. In other words, we find the solutions to the equation $z^2 = 1$. Of course, we already know that the square roots of 1 are 1 and -1, but it will be instructive to utilize our new process and see that it gives the same result. Note that the trigonometric form of 1 is

$$1 = \cos(0) + i\sin(0),$$

so the two square roots of 1 are

$$\sqrt{1}\left[\cos\left(\frac{0+2\pi(0)}{2}\right)+i\,\sin\left(\frac{0+2\pi(0)}{2}\right)\right]=\cos(0)+i\,\sin(0)=1$$

and

$$\sqrt{1}\left[\cos\left(\frac{0+2\pi(1)}{2}\right)+i\,\sin\left(\frac{0+2\pi(1)}{2}\right)\right]=\cos(\pi)+i\,\sin(\pi)=-1$$

as expected.

Progress Check 5.13 (Roots of Unity)

1. Find all solutions to $x^4 = 1$. (The solutions to $x^n = 1$ are called the nth roots of unity, with unity being the number 1.)
2. Find all sixth roots of unity.

Now let's apply our result to find roots of complex numbers other than 1.

Example 5.14 (Roots of Other Complex Numbers)
We will find the solutions to the equation

$$x^4 = -8 + 8\sqrt{3}i.$$

Note that we can write the right hand side of this equation in trigonometric form as

$$-8+8\sqrt{3}i = 16\left(\cos\left(\frac{2\pi}{3}\right)+i\,\sin\left(\frac{2\pi}{3}\right)\right).$$

The fourth roots of $-8+8\sqrt{3}i$ are then

$$\begin{aligned}
x_0 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(0)}{4}\right)+i\,\sin\left(\frac{\frac{2\pi}{3}+2\pi(0)}{4}\right)\right]\\
&= 2\left[\cos\left(\frac{\pi}{6}\right)+i\,\sin\left(\frac{\pi}{6}\right)\right]\\
&= 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\\
&= \sqrt{3}+i,
\end{aligned}$$

$$
\begin{aligned}
x_1 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(1)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(1)}{4}\right)\right]\\
&= 2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]\\
&= 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\
&= -1+\sqrt{3}i,
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(2)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(2)}{4}\right)\right]\\
&= 2\left[\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right]\\
&= 2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\\
&= -\sqrt{3}-i,
\end{aligned}
$$

and

$$
\begin{aligned}
x_3 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)\right]\\
&= 2\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right]\\
&= 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\\
&= 1-\sqrt{3}i.
\end{aligned}
$$

Progress Check 5.15 (Fourth Roots of −256)
Find all fourth roots of −256, that is find all solutions of the equation $x^4 = -256$.

Summary of Section 5.3

In this section, we studied the following important concepts and ideas:

- **DeMoivre's Theorem**. Let $z = r(\cos(\theta) + i\,\sin(\theta))$ be a complex number and n any integer. Then
$$z^n = r^n(\cos(n\theta) + i\,\sin(n\theta)).$$
- **Roots of Complex Numbers**. Let n be a positive integer. The nth roots of the complex number $r\,[\cos(\theta) + i\,\sin(\theta)]$ are given by
$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\,\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$
for $k = 0, 1, 2, \ldots, (n-1)$.

Exercises for Section 5.3

1. Use DeMoivre's Theorem to determine each of the following powers of a complex number. Write the answer in the form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

- ⋆ **(a)** $(2 + 2i)^6$
- ⋆ **(b)** $\left(\sqrt{3} + i\right)^8$
- **(c)** $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$
- **(d)** $2\left(\cos\left(\frac{\pi}{15}\right) + i\,\sin\left(\frac{\pi}{15}\right)\right)^{10}$
- **(e)** $\left(1 + i\sqrt{3}\right)^{-4}$
- **(f)** $(-3 + 3i)^{-3}$

2. In each of the following, determine the indicated roots of the given complex number. When it is possible, write the roots in the form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function. Otherwise, leave the roots in polar form.

- ⋆ **(a)** The two square roots of $16i$.
- **(b)** The two square roots of $2 + 2i\sqrt{3}$.
- ⋆ **(c)** The three cube roots of $5\left(\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right)$.
- **(d)** The five fifth roots of unity.
- **(e)** The four fourth roots of $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$.
- **(f)** The three cube roots of $1 + \sqrt{3}i$.

5.4 The Polar Coordinate System

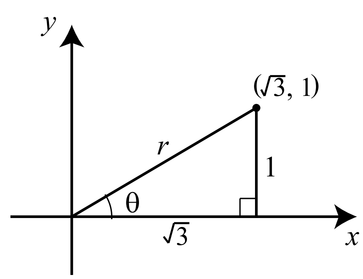
Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How are the polar coordinates of a point in the plane determined?
- How do we convert from polar coordinates to rectangular coordinates?
- How do we convert from rectangular to polar coordinates?
- How do we correctly graph polar equations both by hand and with a calculator?

Beginning Activity

In the diagram to the right, the point with coordinates $(\sqrt{3}, 1)$ has been plotted. Determine the value of r and the angle θ in radians and degrees.



Introduction

In our study of trigonometry so far, whenever we graphed an equation or located a point in the plane, we have used rectangular (or Cartesian [3]) coordinates. The use of this type of coordinate system revolutionized mathematics since it provided the first systematic link between geometry and algebra. Even though the rectangular coordinate system is very important, there are other methods of locating points in the plane. We will study one such system in this section.

Rectangular coordinates use two numbers (in the form of an ordered pair) to determine the location of a point in the plane. These numbers give the position of a

[3] Named after the 17^{th} century mathematician, René Descartes)

point relative to a pair of perpendicular axes. In the beginning activity, to reach the point that corresponds to the ordered pair $\left(\sqrt{3}, 1\right)$, we start at the origin and travel $\sqrt{3}$ units to the right and then travel 1 unit up. The idea of the polar coordinate system is to give a distance to travel and an angle in which direction to travel. We reach the same point as the one given by the rectangular coordinates $\left(\sqrt{3}, 1\right)$ by saying we will travel 2 units at an angle of $30°$ from the x-axis. These values correspond to the values of r and θ in the diagram for the beginning activity. Using the Pythagorean Theorem, we can obtain $r = 2$ and using the fact that $\sin(\theta) = \dfrac{1}{2}$, we see that $\theta = \dfrac{\pi}{6}$ radians or $30°$.

The Polar Coordinate System

For the rectangular coordinate system, we use two numbers, in the form of an ordered pair, to locate a point in the plane. We do the same thing for polar coordinates, but now the first number represents a distance from a point and the second number represents an angle. In the **polar coordinate system**, we start with a point O, called the **pole** and from this point, we draw a horizontal ray (directed half-line) called the **polar axis**. We can then assign polar coordinates (r, θ) to a point P in the plane as follows (see Figure 5.5):

- The number r, called the **radial distance**, is the directed distance from the pole to the point P.
- The number θ, called the **polar angle**, is the measure of the angle from the polar axis to the line segment OP. (Either radians or degrees can be used for the measure of the angle.)

Conventions for Polar Coordinates

- The polar angle θ is considered positive if measured in a counterclockwise direction from the polar axis.
- The polar angle θ is considered negative if measured in a clockwise direction from the polar axis.
- If the radial distance r is positive, then the point P is r units from O along the terminal side of θ.

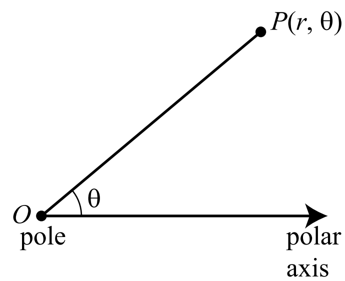


Figure 5.5: Polar Coordinates

- If the radial distance r is negative, then the point P is $|r|$ units from O along the ray in the opposite direction as the terminal side of θ.
- If the radial distance r is zero, then the point P is the point O.

To illustrate some of these conventions, consider the point $P\left(3, \dfrac{4\pi}{3}\right)$ shown on the left in Figure 5.6. (Notice that the circle of radius 3 with center at the pole has been drawn.)

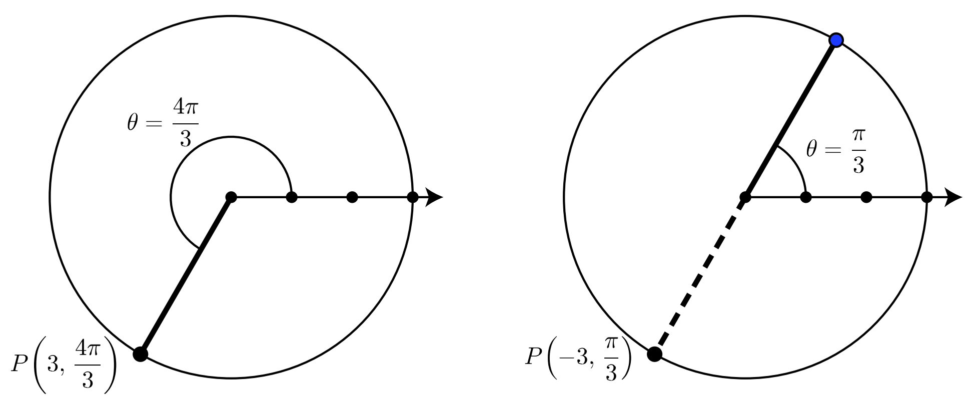


Figure 5.6: A Point with Two Different Sets of Polar Coordinates

The diagram on the right in Figure 5.6 illustrates that this point P also has polar coordinates $P\left(-3, \dfrac{\pi}{3}\right)$. This is because when we use the polar angle $\theta = \dfrac{\pi}{3}$ and the radial distance $r = -3$, the point P is 3 units from the pole along the ray in the opposite direction as the terminal side of θ.

Progress Check 5.16 (Plotting Points in Polar Coordinates)
Since a point with polar coordinates (r, θ) must lie on a circle of radius r with center at the pole, it is reasonable to locate points on a grid of concentric circles and rays whose initial point is at the pole as shown in Figure 5.7. On this polar graph paper, each angle increment is $\dfrac{\pi}{12}$ radians. For example, the point $\left(4, \dfrac{\pi}{6}\right)$ is plotted in Figure 5.7.

Plot the following points with the specified polar coordinates.

$\left(1, \dfrac{\pi}{4}\right)$	$\left(5, \dfrac{\pi}{4}\right)$	$\left(2, \dfrac{\pi}{3}\right)$
$\left(3, \dfrac{5\pi}{4}\right)$	$\left(4, -\dfrac{\pi}{4}\right)$	$\left(4, \dfrac{7\pi}{4}\right)$
$\left(6, \dfrac{5\pi}{6}\right)$	$\left(5, \dfrac{9\pi}{4}\right)$	$\left(-5, \dfrac{5\pi}{4}\right)$

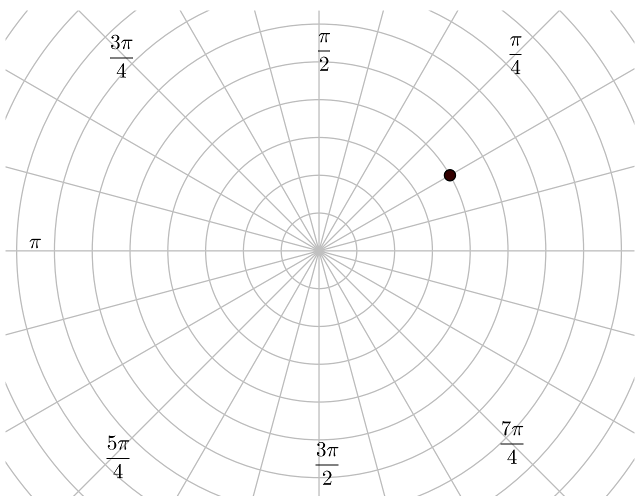


Figure 5.7: Polar Graph Paper

In Progress Check 5.16, we noticed that the polar coordinates $\left(5, \dfrac{\pi}{4}\right)$, $\left(5, \dfrac{9\pi}{4}\right)$, and $\left(-5, \dfrac{5\pi}{4}\right)$ all determined the same point in the plane. This illustrates a major difference between rectangular coordinates and polar coordinates. Whereas each point has a unique representation in rectangular coordinates, a given point can have many different representations in polar coordinates. This is primarily due to the fact that the polar coordinate system uses concentric circles for its grid, and we can start at a point on a circle and travel around the circle and end at the point from which we started. Since one wrap around a circle corresponds to an angle of 2π radians or $360°$, we have the following:

Polar Coordinates of a Point
A point P, other than the pole, determined by the polar coordinates (r, θ) is also determined by the following polar coordinates:

$$\begin{array}{lcc} \text{In radians :} & (r, \theta + k(2\pi)) & (-r, +\theta + (2k+1)\pi) \\ \text{In degrees :} & \left(r, \theta + k\left(360°\right)\right) & \left(-r, \theta + (2k+1)180°\right) \end{array}$$

where k can be any integer.
If the point P is the pole, the its polar coordinates are $(0, \theta)$ for any polar angle θ.

Progress Check 5.17 (Different Polar Coordinates for a Point)
Find four different representations in polar coordinates for the point with polar coordinates $(3, 110°)$. Use a positive value for the radial distance r for two of the representations and a negative value for the radial distance r for the other two representations.

Conversions Between Polar and Rectangular Coordinates

We now have two ways to locate points in the plane. One is the usual rectangular (Cartesian) coordinate system and the other is the polar coordinate system. The rectangular coordinate system uses two distances to locate a point, whereas the polar coordinate system uses a distance and an angle to locate a point. Although these two systems can be studied independently of each other, we can set them up so that there is a relationship between the two types of coordinates. We do this as follows:

- We place the pole of the polar coordinate system at the origin of the rectangular coordinate system.
- We have the polar axis of the polar coordinate system coincide with the positive x-axis of the rectangular coordinate system as shown in Figure 5.8

Using right triangle trigonometry and the Pythagorean Theorem, we obtain the following relationships between the rectangular coordinates (x, y) and the polar

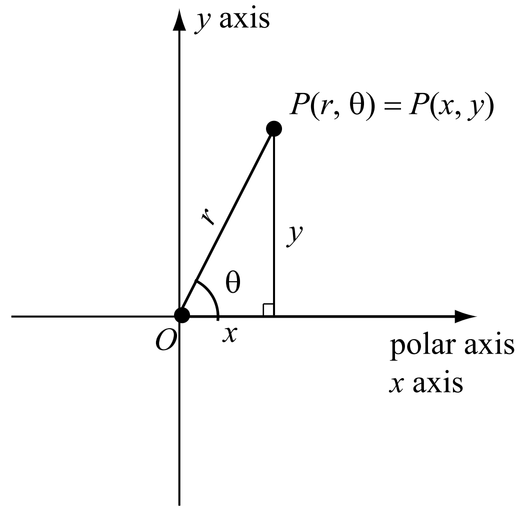


Figure 5.8: Polar and Rectangular Coordinates

coordinates (r, θ):

$$\cos(\theta) = \frac{x}{r} \qquad x = r\cos(\theta)$$
$$\sin(\theta) = \frac{y}{r} \qquad y = r\sin(\theta)$$
$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0 \qquad x^2 + y^2 = r^2$$

Coordinate Conversion
To determine the rectangular coordinates (x, y) of a point whose polar coordinates (r, θ) are known, use the equations

$$x = r\cos(\theta) \qquad y = r\sin(\theta).$$

To determine the polar coordinates (r, θ) of a point whose rectangular coordinates (x, y) are known, use the equation $r^2 = x^2 + y^2$ to determine r and determine an angle θ so that

$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0 \qquad \cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r}.$$

When determining the polar coordinates of a point, we usually choose the positive value for r. We can use an inverse trigonometric function to help determine θ but we must be careful to place θ in the proper quadrant by using the signs of x and y. Note that if $x = 0$, we can use $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$.

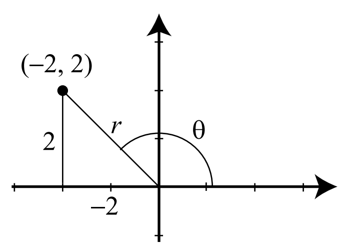
Progress Check 5.18 (Converting from Polar to Rectangular Coordinates)
Determine rectangular coordinates for each of the following points in polar coordinates:

1. $\left(3, \dfrac{\pi}{3}\right)$ **2.** $\left(5, \dfrac{11\pi}{6}\right)$ **3.** $\left(-5, \dfrac{3\pi}{4}\right)$

When we convert from rectangular coordinates to polar coordinates, we must be careful and use the signs of x and y to determine the proper quadrant for the angle θ. In many situations, it might be easier to first determine the reference angle for the angle θ and then use the signs of x and y to determine θ.

Example 5.19 (Converting from Rectangular to Polar Coordinates)
To determine polar coordinates for the point $(-2, 2)$ in rectangular coordinates, we first draw a picture and note that

$$r^2 = (-2)^2 + 2^2 = 8.$$



Since it is usually easier to work with a positive value for r, we will use $r = \sqrt{8}$.

We also see that $\tan(\theta) = \dfrac{3}{-3} = -1$. We can use many different values for θ but to keep it easy, we use θ as shown in the diagram. For the reference angle $\hat{\theta}$, we have $\tan(\hat{\theta}) = 1$ and so $\hat{\theta} = \dfrac{\pi}{4}$. Since $-2 < 0$ and $2 > 0$, θ is in the second quadrant, and we have

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

So the point $(-2, 2)$ in rectangular coordinates has polar coordinates $\left(\sqrt{8}, \dfrac{3\pi}{4}\right)$.

Progress Check 5.20 (Converting from Rectangular to Polar Coordinates)
Determine polar coordinates for each of the following points in rectangular coordinates:

1. $\left(6, 6\sqrt{3}\right)$ **2.** $(0, -4)$ **3.** $(-4, 5)$

In each case, use a positive radial distance r and a polar angle θ with $0 \leq \theta < 2\pi$. An inverse trigonometric function will need to be used for (3).

The Graph of a Polar Equation

The graph an equation on the rectangular coordinate system consists of all points (x, y) that satisfy the equation. The equation can often be written in the form of a function such as $y = f(x)$. In this case, a point (a, b) is on the graph of this function if and only if $b = f(a)$. In a similar manner,

> An equation whose variables are polar coordinates (usually r and θ) is called a **polar equation**. The **graph of a polar equation** is the set of all points whose polar coordinates (r, θ) satisfy the given equation.

An example of a polar equation is $r = 4\sin(\theta)$. For this equation, notice that

- If $\theta = 0$, then $r = 4\sin(0) = 0$ and so the point $(0, 0)$ (in polar coordinates) is on the graph of this equation.
- If $\theta = \dfrac{\pi}{6}$, then $r = 4\sin\left(\dfrac{\pi}{6}\right) = 4 \cdot \dfrac{1}{2} = 2$ and so $\left(2, \dfrac{\pi}{6}\right)$ is on the graph of this equation. (Remember: for polar coordinates, the value of r is the first coordinate.)

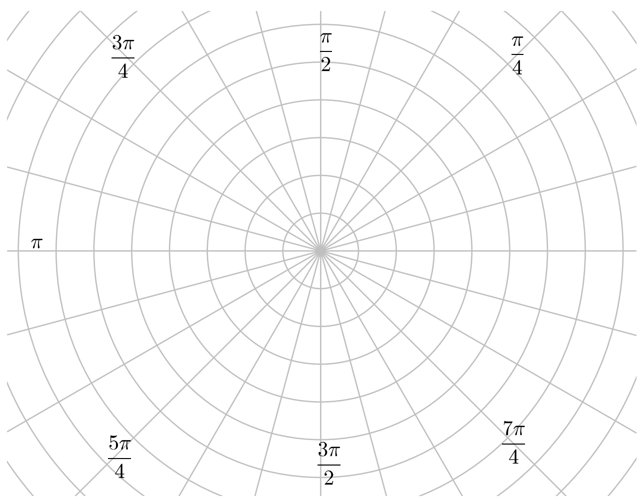
The most basic method for drawing the graph of a polar equation is to plot the points that satisfy the polar equation on polar graph paper as shown in Figure 5.7 and then connect the points with a smooth curve.

Progress Check 5.21 (Graphing a Polar Equation)
The following table shows the values of r and θ for points that are on the graph of the polar equation $r = 4\sin(\theta)$.

$r = 4\sin(\theta)$	θ
0	0
2	$\frac{\pi}{6}$
$2\sqrt{2}$	$\frac{\pi}{4}$
$2\sqrt{3}$	$\frac{\pi}{3}$
4	$\frac{\pi}{2}$

$r = 4\sin(\theta)$	θ
$2\sqrt{3}$	$\frac{2\pi}{3}$
$2\sqrt{2}$	$\frac{3\pi}{4}$
2	$\frac{5\pi}{6}$
0	π

Plot these points on polar graph paper and draw a smooth curve through the points for the graph of the equation $r = 4\sin(\theta)$.



Depending on how carefully we plot the points and how well we draw the curve, the graph in Progress Check 5.21 could be a circle. We can, of course, plot more points. In fact, in Progress Check 5.21, we only used values for θ with $0 \le \theta \le \pi$. The following table shows the values of r and θ for points that are on the graph of the polar equation $r = 4\sin(\theta)$ with $\pi \le \theta \le 2\pi$.

$r = 4\sin(\theta)$	θ
0	π
-2	$\frac{7\pi}{6}$
$-2\sqrt{2}$	$5\frac{\pi}{4}$
$-2\sqrt{3}$	$4\frac{\pi}{3}$
-4	$\frac{3\pi}{2}$

$r = 4\sin(\theta)$	θ
$-2\sqrt{3}$	$\frac{5\pi}{3}$
$-2\sqrt{2}$	$\frac{7\pi}{4}$
-2	$\frac{11\pi}{6}$
0	π

Because of the negative values for r, if we plot these points, we will get the same points we did in Progress Check 5.21. So a good question to ask is, "Do these points really lie on a circle?" We can answer this question by converting the equation $r = 4\sin(\theta)$ into an equivalent equation with rectangular coordinates.

Transforming an Equation from Polar Form to Rectangular Form

The formulas that we used to convert a point in polar coordinates to rectangular coordinates can also be used to convert an equation in polar form to rectangular form. These equations are given in the box on page 329. So let us look at the equation $r = 4\sin(\theta)$ from Progress Check 5.21.

Progress Check 5.22 (Transforming a Polar Equation into Rectangular Form)
We start with the equation $r = 4\sin(\theta)$. We want to transform this into an equation involving x and y. Since $r^2 = x^2 + y^2$, it might be easier to work with r^2 rather than r.

1. Multiply both sides of the equation $r = 4\sin(\theta)$ by r.
2. Now use the equations $r^2 = x^2 + y^2$ and $y = r\sin(\theta)$ to obtain an equivalent equation in x and y.

The graph of the equation the graph of $r = 4\sin(\theta)$ in polar coordinates will be the same as the graph of $x^2 + y^2 = 4y$ in rectangular coordinates. We can now use some algebra from previous mathematics courses to show that this is the graph of a circle. The idea is to collect all terms on the left side of the equation and use completing the square for the terms involving y.

As a reminder, if we have the expression $t^2 + at = 0$, we complete the square by adding $\left(\frac{a}{2}\right)^2$ to both sides of the equation. We will then have a perfect square on the left side of the equation.

$$\begin{aligned} t^2 + at + \left(\frac{a}{2}\right)^2 &= \left(\frac{a}{2}\right)^2 \\ t^2 + at + \frac{a^2}{4} &= \frac{a^2}{4} \\ \left(t + \frac{a}{2}\right)^2 &= \frac{a^2}{4} \end{aligned}$$

So for the equation $x^2 + y^2 = 4y$, we have

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + y^2 - 4y + 4 &= 4 \\ x^2 + (y-2)^2 &= 2^2 \end{aligned}$$

This is the equation (in rectangular coordinates) of a circle with radius 2 and center at the point $(0, 2)$. We see that this is consistent with the graph we obtained in Progress Check 5.21.

Progress Check 5.23 (Transforming a Polar Equation into Rectangular Form)
Transform the equation $r = 6\cos(\theta)$ into an equation in rectangular coordinates and then explain why the graph of $r = 6\cos(\theta)$ is a circle. What is the radius of this circle and what is its center?

The Polar Grid

We introduced polar graph paper in Figure 5.7. Notice that this consists of concentric circles centered at the pole and lines that pass through the pole. These circles and lines have very simple equations in polar coordinates. For example:

- Consider the equation $r = 3$. In order for a point to be on the graph of this equation, it must lie on a circle of radius 3 whose center is the pole. So the graph of this equation is a circle of radius 3 whose center is the pole. We can also show this by converting the equation $r = 3$ to rectangular form as follows:

  $$\begin{aligned} r &= 3 \\ r^2 &= 3^2 \\ x^2 + y^2 &= 9 \end{aligned}$$

  In rectangular coordinates, this is the equation of a circle of radius 3 centered at the origin.

- Now consider the equation $\theta = \dfrac{\pi}{4}$. In order for a point to be on the graph of this equation, the line through the pole and this point must make an angle of $\dfrac{\pi}{4}$ radians with the polar axis. If we only allow positive values for r, the graph will be a ray with initial point at the pole that makes an angle of $\dfrac{\pi}{4}$ with the polar axis. However, if we allow r to be any real number, then we obtain the line through the pole that makes an angle of $\dfrac{\pi}{4}$ radians with the polar axis. We can convert this equation to rectangular coordinates as follows:

$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan(\theta) &= \tan\left(\frac{\pi}{4}\right) \\ \frac{y}{x} &= 1 \\ y &= x \end{aligned}$$

  This is an equation for a straight line through the origin with a slope of 1.

In general:

The Polar Grid

- If a is a positive real number, then the graph of $r = a$ is a circle of radius a whose center is the pole.
- If b is a real number, then the graph of $\theta = b$ is a line through the pole that makes an angle of b radians with the polar axis.

Concluding Remarks

We have studied just a few graphs of polar equations. There are many interesting graphs that can be generated using polar equations that are very difficult to accomplish in rectangular coordinates. Since the polar coordinate system is based on concentric circles, it should not be surprising that circles with center at the pole would have "simple" equations like $r = a$.

In Progress Checks 5.21 and 5.23, we saw polar equations whose graphs were circles with centers not at the pole. These were special cases of the following:

Polar Equations Whose Graphs Are Circles
If a is a positive real number, then

- The graph of $r = 2a\sin(\theta)$ is a circle of radius a with center at the point $(0, a)$ in rectangular coordinates or $\left(a, \dfrac{\pi}{2}\right)$ in polar coordinates.
- The graph of $r = 2a\cos(\theta)$ is a circle of radius a with center at the point $(a, 0)$ in rectangular coordinates or $(a, 0)$ in polar coordinates.

We will explore this and the graphs of other polar equations in the exercises.

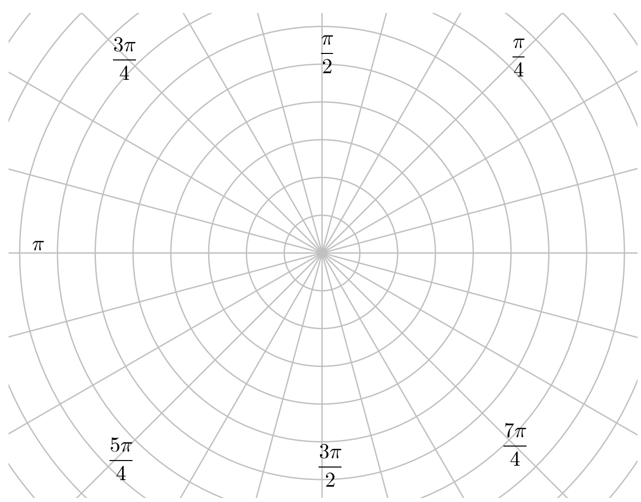
Exercises for Section 5.4

⋆ 1. Plot the following points with the specified polar coordinates.

$$\left(7, \frac{\pi}{6}\right) \qquad \left(3, \frac{3\pi}{4}\right) \qquad \left(2, \frac{-\pi}{3}\right)$$

$$\left(3, \frac{7\pi}{4}\right) \qquad \left(5, -\frac{\pi}{4}\right) \qquad \left(4, \frac{11\pi}{4}\right)$$

$$\left(6, \frac{11\pi}{6}\right) \qquad \left(-3, \frac{2\pi}{3}\right) \qquad \left(-5, \frac{5\pi}{6}\right)$$



2. For each of the following points in polar coordinates, determine three different representations in polar coordinates for the point. Use a positive value for the radial distance r for two of the representations and a negative value for the radial distance r for the other representation.

⋆ **(a)** $(5, 30°)$

⋆ **(b)** $(4, 100°)$

(c) $(-2, 50°)$

(d) $(7, -60°)$

3. For each of the following points in polar coordinates, determine three different representations in polar coordinates for the point. Use a positive value for the radial distance r for two of the representations and a negative value for the radial distance r for the other representation. **Note**: The angles are measured in radians.

⋆ **(a)** $\left(5, \frac{\pi}{6}\right)$

⋆ **(b)** $\left(4, \frac{5\pi}{9}\right)$

(c) $\left(-2, \frac{5\pi}{18}\right)$

(d) $\left(7, -\frac{\pi}{3}\right)$

4. Determine rectangular coordinates for each of the following points in polar coordinates:

⋆ **(a)** $\left(10, \frac{2\pi}{3}\right)$

(b) $\left(8, \frac{7\pi}{6}\right)$

⋆ **(c)** $\left(-5, \frac{5\pi}{4}\right)$

(d) $\left(10, -\frac{2\pi}{3}\right)$

(e) $\left(3, \frac{5\pi}{3}\right)$

(f) $\left(6, -\frac{\pi}{6}\right)$

5. Determine polar coordinates for each of the following points in rectangular coordinates. Use a positive radial distance r and a polar angle with $0 \leq \theta < 2\pi$. When necessary, use an inverse trigonometric function and round the angle (in radians) to the nearest thousandth.

⋆ **(a)** $\left(\frac{-5\sqrt{3}}{2}, \frac{5}{2}\right)$

⋆ **(b)** $(3, 5)$

(c) $\left(\sqrt{2}, -\sqrt{2}\right)$

(d) $(-3, -4)$

6. Convert each of the following polar equations into a rectangular equation. If possible, write the rectangular equation with y as a function of x.

⋆ **(a)** $r = 5$

⋆ **(b)** $\theta = \dfrac{\pi}{3}$

(c) $r = 8\cos(\theta)$

⋆ **(d)** $r = 1 - \sin(\theta)$

(e) $r^2 \sin(2\theta) = 1$

(f) $r = 1 - 2\cos(\theta)$

(g) $r = \dfrac{3}{\sin(\theta) + 4\cos(\theta)}$

7. Convert each of the following rectangular equations into a polar equation. If possible, write the polar equation with r as a function of θ.

(**a**) $x^2 + y^2 = 36$

⋆ **(b)** $y = 4$

(c) $x = 7$

(d) $x^2 - 6x + y^2 = 0$

⋆ **(e)** $x + y = 4$

(f) $y = x^2$

8. Let a be a positive real number.

(**a**) Convert the polar equation $r = 2a\sin(\theta)$ to rectangular coordinates and then explain why the graph of this equation is a circle. What is the radius of the circle and what is the center of the circle in rectangular coordinates?

(**b**) Convert the polar equation $r = 2a\cos(\theta)$ to rectangular coordinates and then explain why the graph of this equation is a circle. What is the radius of the circle and what is the center of the circle in rectangular coordinates?

Appendix A

Answers for the Progress Checks

Section 1.1

Progress Check 1.1

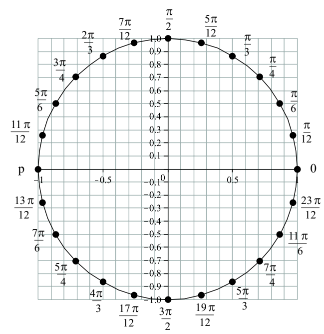
1. Some positive numbers that are wrapped to the point $(-1, 0)$ are $\pi, 3\pi, 5\pi$.
   Some negative numbers that are wrapped to the point $(-1, 0)$ are $-\pi, -3\pi, -5\pi$.

2. The numbers that get wrapped to $(-1, 0)$ are the odd integer multiples of π.

3. Some positive numbers that are wrapped to the point $(0, 1)$ are $\dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$.
   Some negative numbers that are wrapped to the point $(0, 1)$ are $-\dfrac{\pi}{2}, -\dfrac{5\pi}{2}, -\dfrac{9\pi}{2}$.

4. Some positive numbers that are wrapped to the point $(0, -1)$ are $\dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \dfrac{11\pi}{2}$.
   Some negative numbers that are wrapped to the point $(0, 1)$ are $-\dfrac{3\pi}{2}, -\dfrac{7\pi}{2}, -\dfrac{11\pi}{2}$.

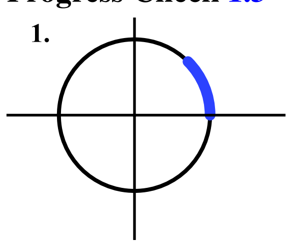
Progress Check 1.2



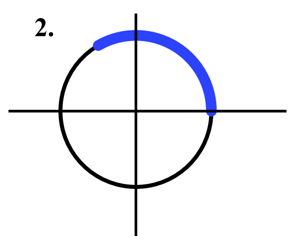
1. For $t = \dfrac{\pi}{3}$, the point is approximately $(0.5, 0.87)$.
2. For $t = \dfrac{2\pi}{3}$, the point is approximately $(-0.5, 0.87)$.
3. For $t = \dfrac{4\pi}{3}$, the point is approximately $(-0.5, -0.87)$.
4. For $t = \dfrac{5\pi}{3}$, the point is approximately $(0.5, -0.87)$.
5. For $t = \dfrac{\pi}{4}$, the point is approximately $(0.71, 0.71)$.
6. For $t = \dfrac{7\pi}{4}$, the point is approximately $(0.71, -0.71)$.

Progress Check 1.3

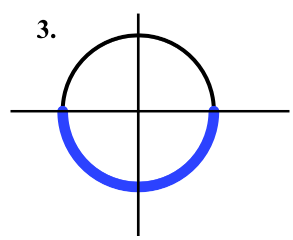
1.



2.



3.



Progress Check 1.4

1. We substitute $y = \dfrac{1}{2}$ into $x^2 + y^2 = 1$.

$$\begin{aligned} x^2 + \left(\frac{1}{2}\right)^2 &= 1 \\ x^2 &= \frac{3}{4} \\ x &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

The two points are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

1. We substitute $x = \frac{\sqrt{5}}{4}$ into $x^2 + y^2 = 1$.

$$\left(\frac{\sqrt{5}}{4}\right)^2 + y^2 = 1$$
$$y^2 = \frac{11}{16}$$
$$y = \pm\frac{\sqrt{11}}{4}$$

The two points are $\left(\frac{\sqrt{5}}{4}, \frac{\sqrt{11}}{4}\right)$ and $\left(\frac{\sqrt{5}}{4}, -\frac{\sqrt{11}}{4}\right)$.

Section 1.2

Progress Check 1.5

1. $\cos\left(\frac{\pi}{2}\right) = 0$
$\sin\left(\frac{\pi}{2}\right) = 1.$

2. $\cos\left(\frac{3\pi}{2}\right) = 0$
$\sin\left(\frac{3\pi}{2}\right) = -1.$

3. $\cos(0) = 1$
$\sin(0) = 0.$

4. $\cos\left(-\frac{\pi}{2}\right) = 0$
$\sin\left(-\frac{\pi}{2}\right) = -1.$

5. $\cos(2\pi) = 1$
$\sin(2\pi) = 1.$

6. $\cos(-\pi) = -1$
$\sin(-\pi) = 0.$

Progress Check 1.6

1. $\cos(1) \approx 0.5403,$
$\sin(1) \approx 0.8415.$

2. $\cos(2) \approx -0.4161$
$\sin(2) \approx 0.9093.$

3. $\cos(-4) \approx -0.6536$
$\sin(-4) \approx 0.7568$.

4. $\cos(5.5) \approx 0.7807$
$\sin(5.5) \approx -0.7055$.

5. $\cos(15) \approx -0.7597$
$\sin(15) \approx 0.6503$.

6. $\cos(-15) \approx -0.7597$
$\sin(-15) \approx -0.6503$.

Progress Check 1.7

1. Since we can wrap any number onto the unit circle, we can always find the terminal point of an arc that corresponds to any number. So the cosine of any real number is defined and the domain of the cosine function is the set of all of the real numbers.

2. For the same reason as for the cosine function, the domain of the sine function is the set of all real numbers.

3. On the unit circle, the largest x-coordinate a point can have is 1 and the smallest x-coordinate a point can have is -1. Since the output of the cosine function is the x-coordinate of a point on the unit circle, the range of the cosine function is the closed interval $[-1, 1]$. That means $-1 \leq \cos(t) \leq 1$ for any real number t.

4. On the unit circle, the largest y-coordinate a point can have is 1 and the smallest y-coordinate a point can have is -1. Since the output of the sine function is the y-coordinate of a point on the unit circle, the range of the sine function is the closed interval $[-1, 1]$. That means $-1 \leq \sin(t) \leq 1$ for any real number t.

Progress Check 1.8

1. If $\dfrac{\pi}{2} < t < \pi$, then the terminal point of the arc t is in the second quadrant and so $\cos(t) < 0$ and $\sin(t) > 0$.

2. If $\pi < t < \dfrac{3\pi}{2}$, then the terminal point of the arc t is in the third quadrant and so $\cos(t) < 0$ and $\sin(t) < 0$.

3. If $\dfrac{3\pi}{2} < t < 2\pi$, then the terminal point of the arc t is in the fourth quadrant and so $\cos(t) > 0$ and $\sin(t) < 0$.

4. If $\dfrac{5\pi}{2} < t < 3\pi$, then the terminal point of the arc t is in the second quadrant and so $\cos(t) < 0$ and $\sin(t) > 0$.

5. Note that $\cos(t) = 0$ at $t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$. Since $\cos(t)$ is the x-coordinate of the terminal point of the arc t, the previous response shows that $\cos(t)$ is positive when t is in one of the intervals $\left[0, \dfrac{\pi}{2}\right)$ or $\left(\dfrac{3\pi}{2}, 2\pi\right]$.

6. Note that $\sin(t) = 0$ at $t = 0$ and $t = \pi$. Since $\sin(t)$ is the y-coordinate of the terminal point of the arc t, the previous response shows that $\sin(t)$ is positive when t is in the interval $(0, \pi)$.

7. Note that $\cos(t) = 0$ at $t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$. Since $\cos(t)$ is the x-coordinate of the terminal point of the arc t, the previous response shows that $\cos(t)$ is negative when t is in the interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

8. Note that $\sin(t) = 0$ at $t = \pi$ and $t = 2\pi$. Since $\sin(t)$ is the y-coordinate of the terminal point of the arc t, the previous response shows that $\sin(t)$ is positive when t is in the interval $(\pi, 2\pi)$.

Progress Check 1.9

1. Since $0 < \dfrac{\pi}{5} < \dfrac{\pi}{2}$, the terminal point of the arc $\dfrac{\pi}{5}$ is in the first quadrant. Therefore, $\cos\left(\dfrac{\pi}{5}\right)$ is positive.

2. Using the information about t in (1), $\sin\left(\dfrac{\pi}{5}\right)$ is positive.

3. We can write $\dfrac{\pi}{2}$ as $\dfrac{4\pi}{8}$ and π as $\dfrac{8\pi}{8}$, so $\dfrac{\pi}{2} < \dfrac{5\pi}{8} < \pi$. This puts the terminal point of the arc $\dfrac{5\pi}{8}$ in the second quadrant. Therefore, $\cos\left(\dfrac{5\pi}{8}\right)$ is negative.

4. Using the information about t in (3), $\sin\left(\dfrac{5\pi}{8}\right)$ is positive.

5. We can write $-\dfrac{\pi}{2}$ as $\dfrac{-8\pi}{16}$ and $-\pi$ as $\dfrac{-16\pi}{16}$, so $-\pi < \dfrac{-9\pi}{16} < -\dfrac{\pi}{2}$. This puts the terminal point of the arc $\dfrac{-9\pi}{16}$ in the third quadrant. Therefore, $\cos\left(\dfrac{-9\pi}{16}\right)$ is negative.

6. Using the information about t in (5), $\sin\left(\dfrac{-9\pi}{16}\right)$ is negative.

7. We can write -2π as $\dfrac{-24\pi}{12}$ and $-\dfrac{5\pi}{2}$ as $\dfrac{-30\pi}{12}$, so $-\dfrac{5\pi}{2} < \dfrac{-25\pi}{12} < -2\pi$. This puts the terminal point of the arc $\dfrac{-25\pi}{12}$ in the fourth quadrant. Therefore, $\cos\left(\dfrac{-25\pi}{12}\right)$ is positive.

8. Using the information about the arc t in (7), $\sin\left(\dfrac{-25\pi}{12}\right)$ is negative.

Progress Check 1.10

Any point on the unit circle satisfies the equation $x^2+y^2 = 1$. Since $(\cos(t), \sin(t))$ is a point on the unit circle, it follows that $(\cos(t))^2 + (\sin(t))^2 = 1$ or

$$\cos^2(t) + \sin^2(t) = 1.$$

Progress Check 1.12

1. Since $\cos(t) = \dfrac{1}{2}$, we can use the Pythagorean Identity to obtain

$$\begin{aligned}\left(\frac{1}{2}\right)^2 + \sin^2(t) &= 1\\ \frac{1}{4} + \sin^2(t) &= 1\\ \sin^2(t) &= \frac{3}{4}\\ \sin(t) &= \pm\sqrt{\frac{3}{4}}\end{aligned}$$

Notice that we cannot determine the sign of $\sin(t)$ using only the Pythagorean Identity. We need further information about the arc t. In this case, we are given that the terminal point of the arc t is in the fourth quadrant, and hence, $\sin(t) < 0$. Consequently,

$$\sin(t) = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}.$$

2. Since $\sin(t) = -\dfrac{2}{3}$, we can use the Pythagorean Identity to obtain

$$\begin{aligned}
\cos^2(t) + \left(-\frac{2}{3}\right)^2 &= 1 \\
\cos^2(t) + \frac{4}{9} &= 1 \\
\cos^2(t) &= \frac{5}{9} \\
\cos(t) &= \pm\sqrt{\frac{5}{9}}
\end{aligned}$$

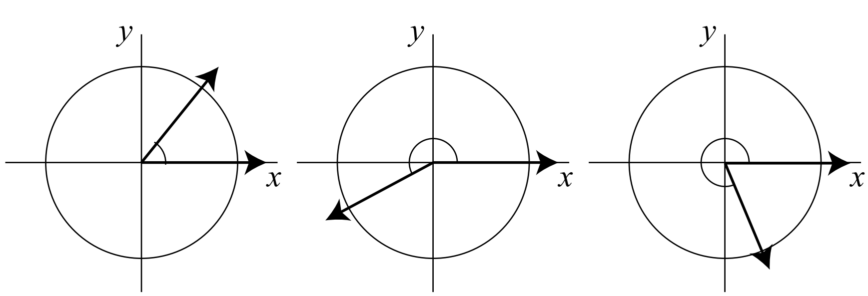
Once again, we need information about the arc t to determine the sign of $\cos(t)$. In this case, we are given that $\pi < t < \dfrac{3\pi}{2}$. Hence, the terminal point of the arc t is in the third quadrant and so, $\cos(t) < 0$. Therefore,

$$\cos(t) = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}.$$

Section 1.3

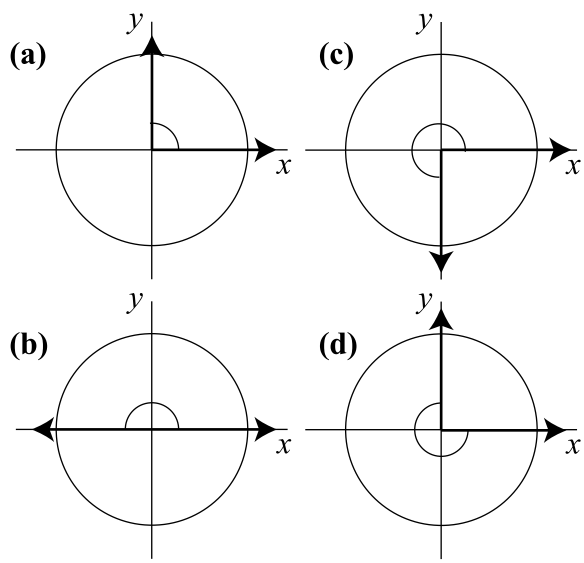
Progress Check 1.13

These graphs show positive angles in standard position. The one on the left has its terminal point in the first quadrant, the one in the middle has its terminal point in the third quadrant, and the one on the right has its terminal point in the fourth quadrant.



Progress Check 1.14

1.



2. (**a**) 90°
(**b**) 180°
(**c**) 270°
(**d**) −270°

Progress Check 1.15

Angle in radians	Angle in degrees	Angle in radians	Angle in degrees
0	0°	$\dfrac{7\pi}{6}$	210°
$\dfrac{\pi}{6}$	30°	$\dfrac{5\pi}{4}$	225°
$\dfrac{\pi}{4}$	45°	$\dfrac{4\pi}{3}$	240°
$\dfrac{\pi}{3}$	60°	$\dfrac{3\pi}{2}$	270°
$\dfrac{\pi}{2}$	90°	$\dfrac{5\pi}{3}$	300°

Angle in radians	Angle in degrees
$\dfrac{2\pi}{3}$	$120°$
$\dfrac{3\pi}{4}$	$135°$
$\dfrac{5\pi}{6}$	$150°$
π	$180°$

Angle in radians	Angle in degrees
$\dfrac{7\pi}{4}$	$315°$
$\dfrac{11\pi}{6}$	$330°$
2π	$360°$

Progress Check 1.16

Using a calculator, we obtain the following results correct to ten decimal places.

- $\cos(1) \approx 0.5403023059$,
  $\sin(1) \approx 0.8414709848$.
- $\cos(2) \approx -0.4161468365$
  $\sin(2) \approx 0.9092974268$.
- $\cos(-4) \approx -0.6536436209$
  $\sin(-4) \approx 0.7568024953$.
- $\cos(-15) \approx -0.7596879129$
  $\sin(-15) \approx -0.6502878402$.

The difference between these values and those obtained in Progress Check 1.6 is that these values are correct to 10 decimal places (and the others are correct to 4 decimal places). If we round off each of the values above to 4 decimal places, we get the same results we obtained in Progress Check 1.6.

Section 1.4

Progress Check 1.17

1. Use the formula $s = r\theta$.

$$s = r\theta = (10\text{ft})\frac{\pi}{2}$$
$$s = 5\pi$$

The arc length is 5π feet.

2. Use the formula $s = r\theta$.

$$s = r\theta = (20\text{ft})\frac{\pi}{2}$$
$$s = 10\pi$$

The arc length is 10π feet.

3. First convert 22° to radians. So $\theta = 22° \times \left(\frac{\pi \text{ rad}}{180°}\right) = \frac{11\pi}{90}$, and

$$s = r\theta = (3\text{ft})\frac{11\pi}{90}$$
$$s = \frac{11\pi}{30}$$

The arc length is $\frac{11\pi}{30}$ feet or about 1.1519 feet.

Progress Check 1.18

1. We see that

$$\omega = 40\frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{\text{rev}}$$
$$\omega = 80\pi\frac{\text{rad}}{\text{min}}$$

2. The result from part (a) gives

$$v = r\left(\frac{\theta}{t}\right) = r\omega$$
$$v = (3 \text{ ft}) \times 80\pi\frac{\text{rad}}{\text{min}}$$
$$v = 240\pi\frac{\text{ft}}{\text{min}}$$

3. We now convert feet per minute to feet per second.

$$v = 240\pi\frac{\text{ft}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}}$$
$$v = 4\pi\frac{\text{ft}}{\text{sec}} \approx 12.566\frac{\text{ft}}{\text{sec}}$$

Progress Check 1.20

1. One revolution corresponds to 2π radians. So
$$\omega = \frac{2\pi \text{ rad}}{24 \text{ hr}} = \frac{\pi}{12} \frac{\text{rad}}{\text{hr}}.$$

2. To determine the linear velocity, we use the formula $v = r\omega$.
$$\begin{aligned} v = r\omega &= (3959 \text{ mi}) \left(\frac{\pi}{12} \frac{\text{rad}}{\text{hr}} \right) \\ &= \frac{3959\pi}{12} \frac{\text{mi}}{\text{hr}} \end{aligned}$$
The linear velocity is approximately 1036.5 miles per hour.

3. To determine the linear velocity, we use the formula $v = r\omega$.
$$\begin{aligned} v = r\omega &= (2800 \text{ mi}) \left(\frac{\pi}{12} \frac{\text{rad}}{\text{hr}} \right) \\ &= \frac{2800\pi}{12} \frac{\text{mi}}{\text{hr}} \end{aligned}$$
The linear velocity is approximately 733.04 miles per hour. To convert this to feet per second, we use the facts that there are 5280 feet in one mile, 60 minutes in an hour, and 60 seconds in a minute. So
$$\begin{aligned} v &= \left(\frac{2800\pi}{12} \frac{\text{mi}}{\text{hr}} \right) \left(\frac{5280 \text{ ft}}{1 \text{ mi}} \right) \left(\frac{1 \text{ hr}}{60 \text{ min}} \right) \left(\frac{1 \text{ min}}{60 \text{ sec}} \right) \\ &= \frac{(2800\pi)(5280)}{12 \cdot 60 \cdot 60} \frac{\text{ft}}{\text{sec}} \end{aligned}$$
So the linear velocity is approximately 1075.1 feet per second.

Section 1.5

Progress Check 1.21

1. $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

2. $\cos\left(\frac{7\pi}{6}\right) - \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$.

3. $\cos\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$.

Progress Check 1.22

1. $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

2. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

3. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

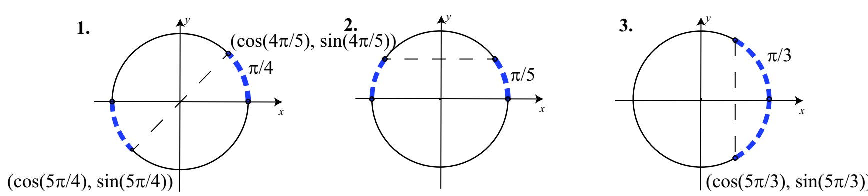
Progress Check 1.23

As shown in the following diagram:

1. The reference arc is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

2. The reference arc is $\pi - \frac{4\pi}{5} = \frac{\pi}{5}$.

3. The reference arc is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$.



Progress Check 1.24

1. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

2. $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

3. $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Progress Check 1.25

1. The terminal point of $t = -\frac{\pi}{6}$ is in the fourth quadrant and the reference arc for $t = -\frac{\pi}{6}$ is $\hat{t} = \frac{\pi}{6}$. So

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}.$$

2. The terminal point of $t = -\frac{2\pi}{3}$ is in the third quadrant and the reference arc for $t = -\frac{2\pi}{3}$ is $\hat{t} = \frac{\pi}{3}$. So

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \text{and} \quad \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

3. The terminal point of $t = -\frac{5\pi}{4}$ is in the second quadrant and the reference arc for $t = -\frac{5\pi}{4}$ is $\hat{t} = \frac{\pi}{4}$. So

$$\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Progress Check 1.27

1. We know that $\pi - t$ is in the second quadrant with reference arc t. So

$$\cos(\pi - t) = -\cos(t) = -\frac{\sqrt{5}}{3}.$$

2. The arc $\pi + t$ is in the third quadrant with reference arc t. So

$$\sin(\pi + t) = -\sin(t) = -\frac{2}{3}.$$

3. The arc $\pi + t$ is in the third quadrant with reference arc t. So

$$\cos(\pi + t) = -\cos(t) = -\frac{\sqrt{5}}{3}.$$

4. The arc $2\pi - t$ is in the fourth quadrant with reference arc t. So

$$\sin(2\pi - t) = -\sin(t) = -\frac{2}{3}.$$

Section 1.6

Progress Check 1.28

1. Since $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$, $\tan(t)$ positive when both $\sin(t)$ and $\cos(t)$ have the same sign. So $\tan(t) > 0$ in the first and third quadrants.

2. We see that $\tan(t)$ negative when$\sin(t)$ and $\cos(t)$ have opposite signs. So $\tan(t) < 0$ in the second and fourth quadrants.

3. $\tan(t)$ will be zero when $\sin(t) = 0$ and $\cos(t) \neq 0$. So $\tan(t) = 0$ when the terminal point of t is on the x-axis. That is, $\tan(t) = 0$ when $t = k\pi$ for some integer k.

4. Following is a completed version of Table 1.4.

t	$\cos(t)$	$\sin(t)$	$\tan(t)$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	1	undefined

Progress Check1.29

1.

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1.$$
$$\tan\left(\frac{5\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}.$$

2. We first use the Pythagorean Identity.

$$\begin{aligned}
\cos^2(t) + \sin^2(t) &= 1 \\
\cos^2(t) + \left(\frac{1}{3}\right)^2 &= 1 \\
\cos^2(t) &= \frac{8}{9}
\end{aligned}$$

Since $\sin(t) > 0$ and $\tan(t) < 0$, we conclude that the terminal point of t must be in the second quadrant, and hence, $\cos(t) < 0$. Therefore,

$$\begin{aligned}
\cos(t) &= -\frac{\sqrt{8}}{3} \\
\tan(t) &= \frac{\frac{1}{3}}{-\frac{\sqrt{8}}{3}} = -\frac{1}{\sqrt{8}}
\end{aligned}$$

Progress Check 1.30

1.

$$\begin{aligned}
\sec\left(\frac{7\pi}{4}\right) &= \frac{1}{\cos\left(\frac{7\pi}{4}\right)} \\
&= \frac{1}{\cos\left(\frac{\pi}{4}\right)} \\
&= \frac{2}{\sqrt{2}} = \sqrt{2}
\end{aligned}$$

2.

$$\begin{aligned}
\csc\left(\frac{-\pi}{4}\right) &= \frac{1}{\sin\left(\frac{-\pi}{4}\right)} \\
&= \frac{1}{\sin\left(-\frac{\pi}{4}\right)} \\
&= -\frac{2}{\sqrt{2}} = -\sqrt{2}
\end{aligned}$$

3. $\tan\left(\frac{7\pi}{8}\right) \approx -0.4142$

4.

$$\begin{aligned}
\cot\left(\frac{4\pi}{3}\right) &= \cot\left(\frac{\pi}{3}\right) \\
&= \frac{1}{\tan\left(\frac{\pi}{3}\right)} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

5. $\csc(5) = \dfrac{1}{\sin(5)} \approx -1.0428$

Progress Check 1.31

1. If $\cos(x) = \dfrac{1}{3}$ and $\sin(x) < 0$, we use the Pythagorean Identity to determine that $\sin(x) = -\dfrac{\sqrt{8}}{3}$. We can then determine that

$$\tan(x) = -\sqrt{8} \qquad \csc(x) = -\frac{3}{\sqrt{8}} \qquad \cot(x) = -\frac{1}{\sqrt{8}}$$

2. If $\sin(x) = -0.7$ and $\tan(x) > 0$, we can use the Pythagorean Identity to obtain

$$\begin{aligned} \cos^2(x) + (-0.7)^2 &= 1 \\ \cos^2(x) &= 0.51 \end{aligned}$$

Since we are also given that $\tan(x) > 0$, we know that the terminal point of x is in the third quadrant. Therefore, $\cos(x) < 0$ and $\cos(x) = -\sqrt{0.51}$. Hence,

$$\begin{aligned} \tan(x) &= \frac{-0.7}{-\sqrt{0.51}} \\ \cot(x) &= \frac{\sqrt{0.51}}{0.7} \end{aligned}$$

3. We can use the definition of $\tan(x)$ to obtain

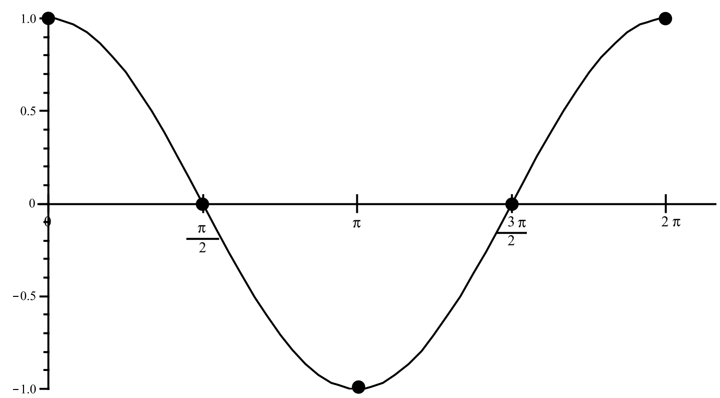
$$\begin{aligned} (\tan(x))(\cos(x)) &= \frac{\sin(x)}{\cos(x)} \cdot \cos(x) \\ &= \sin(x) \end{aligned}$$

So $\tan(x)\cos(x) = \sin(x)$, but it should be noted that this equation is only valid for those values of x for which $\tan(x)$ is defined. That is, this equation is only valid if x is not an integer multiple of π.

Section 2.1

Progress Check 2.1

Not all of the points are plotted, but the following is a graph of one complete period of $y = \cos(t)$ for $0 \le t \le 2\pi$.



Progress Check 2.2

1. The difference is that the graph in Figure 2.2 shows three complete periods of $y = \cos(t)$ over the interval $[-2\pi, 4\pi]$.

2. The graph of $y = \cos(t)$ has t-intercepts at $t = -\dfrac{3\pi}{2}$, $t = -\dfrac{\pi}{2}$, $t = \dfrac{\pi}{2}$, $t = \dfrac{3\pi}{2}$, $t = \dfrac{5\pi}{2}$, and $t = \dfrac{7\pi}{2}$.

3. The maximum value of $y = \cos(t)$ is 1. The graph attains this maximum at $t = -2\pi$, $t = 0$, $t = 2\pi$, and $t = 4\pi$.

4. The minimum value of $y = \cos(t)$ is -1. The graph attains this minimum at $t = -\pi$, $t = \pi$, and $t = 3\pi$.

Progress Check 2.4

- The graph of $y = \sin(t)$ has t-intercepts of $t = 0$, $t = \pi$, and $t = 2\pi$ in the interval $[0, 2\pi]$.

- If we add the period of 2π to each of these t-intercepts and subtract the period of 2π from each of these t-intercepts, we see that the graph of $y = \sin(t)$ has t-intercepts of $t = -2\pi$, $t = -\pi$, $t = 0$, $t = \pi$, $t = 2\pi$, $t = 3\pi$, and $t = 4\pi$ in the interval $[-2\pi, 4\pi]$.

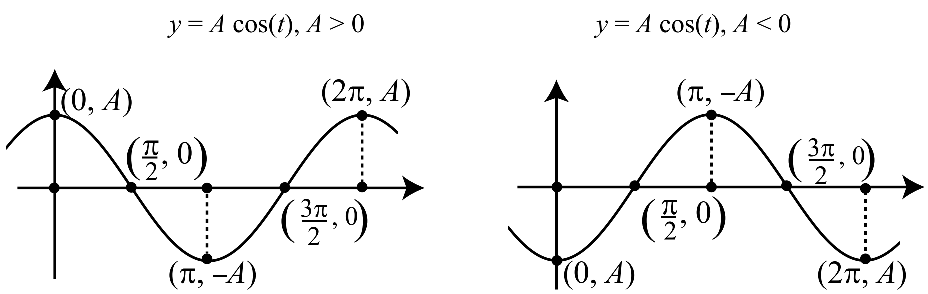
We can determine other t-intercepts of $y = \sin(t)$ by repeatedly adding or subtracting the period of 2π. For example, there is a t-intercept at:

- $t = 3\pi + 2\pi = 5\pi$;

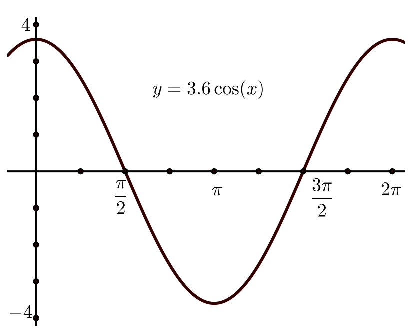
- $t = 5\pi + 2\pi = 7\pi$.

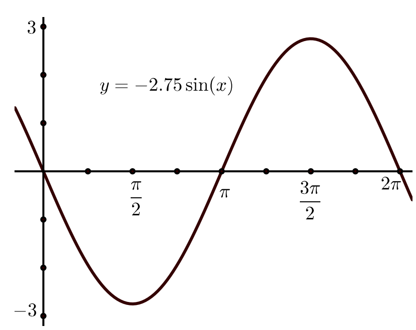
However, if we look more carefully at the graph of $y = \sin(t)$, we see that the t-intercepts are spaced π units apart. This means that we can say that $t = 0 + k\pi$, where k is some integer, is a t-intercept of $y = \sin(t)$.

Progress Check 2.6



Progress Check 2.7

1. 

2. 

Section 2.2

Progress Check 2.9

1. (a) For $y = 3\cos\left(\dfrac{1}{3}t\right)$, the amplitude is 3 and the period is $\dfrac{2\pi}{\dfrac{1}{3}} = 6\pi$.

(**b**) For $y = -2\sin\left(\frac{\pi}{2}t\right)$, the amplitude is 2 and the period is $\dfrac{2\pi}{\frac{\pi}{2}} = 4$.

2. From the graph, the amplitude is 2.5 and the period is 2. Using a cosine function, we have $A = 2.5$ and $\dfrac{2\pi}{B} = 2$. Solving for B gives $B = \pi$. So an equation is $y = 2.5\cos(\pi t)$.

Progress Check 2.11

1. (**a**) For $y = 3.2\left(\sin(t - \frac{\pi}{3}\right)$, the amplitude is 3.2 and the phase shift is $\dfrac{\pi}{3}$.

(**b**) For $y = 4\cos\left(t + \frac{\pi}{6}\right)$, notice that $y = 4\cos\left(t - \left(-\frac{\pi}{6}\right)\right)$. So the amplitude is 4 and the phase shift is $-\dfrac{\pi}{6}$.

2. There are several possible equations for this sinusoid. Some of these equations are:

$$y = 3\sin\left(t + \frac{3\pi}{4}\right) \qquad y = 3\cos\left(t + \frac{\pi}{4}\right)$$
$$y = -3\sin\left(t - \frac{\pi}{4}\right) \qquad y = -3\cos\left(t - \frac{3\pi}{4}\right)$$

A graphing utility can be used to verify that any of these equations produce the given graph.

Progress Check 2.14

1. The amplitude is 6.3.

2. The period is $\dfrac{2\pi}{50\pi} = \dfrac{1}{25}$.

3. We write $y = 6.3\cos(50\pi(t - (-0.01))) + 2$ and see that the phase shift is -0.01 or 0.01 units to the left.

4. The vertical shift is 2.

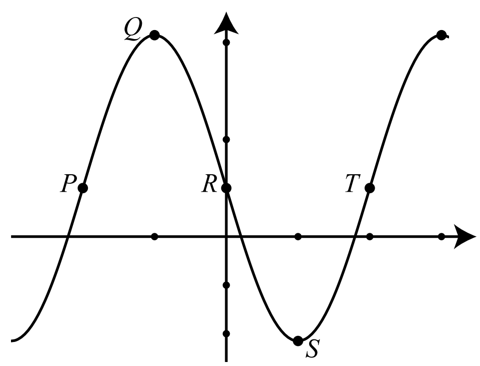
5. Because we are using a cosine and the phase shift is -0.01, we can use -0.01 as the t-coordinate of Q. The y-coordinate will be the vertical shift plus the amplitude. So the y-coordinate is 8.3. Point Q has coordinates $(-0.01, 8.3)$.

6. We now use the fact that the horizontal distance between P and Q is one-quarter of a period. Since the period is $\dfrac{1}{25} = 0.04$, we see that one-quarter of a period is 0.01. The point P also lies on the center line, which is $y = 2$. So the coordinates of P are $(-0.02, 2)$.

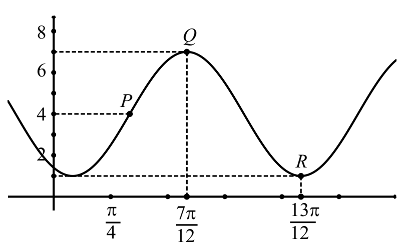
We now use the fact that the horizontal distance between Q and R is one-quarter of a period. The point R is on the center line of the sinusoid and so R has coordinates $(0, 2)$.

The point S is a low point on the sinusoid. So its y-coordinate will be D minus the amplitude, which is $2 - 6.3 = -4.3$. Using the fact that the horizontal distance from R to S is one-quarter of a period, the coordinates of S are $(0.01, -4.3)$. Since the point T is on the center line and the horizontal distance from S to T is one-quarter of a period, the coordinates of T are $(0.03, 2)$.

7. We will use a viewing window that is one-quarter of a period to the left of P and one-quarter of a period to the right of T. So we will use $-0.03 \le t \le 0.03$. Since the maximum value is 8.3 and the minimum value is -4.3, we will use $-5 \le y \le 9$.



Progress Check 2.15



1. The coordinates of Q are $\left(\dfrac{7\pi}{12}, 7\right)$ and the coordinates of R are $\left(\dfrac{13\pi}{12}, 1\right)$. So two times the amplitude is $7 - 1 = 6$ and the amplitude is 3.

2. We add the amplitude to the lowest y-value to determine D. This gives $D = 1 + 3 = 4$ and the center line is $y = 4$.

3. The horizontal distance between Q and R is $\dfrac{13\pi}{12} - \dfrac{7\pi}{12} = \dfrac{6\pi}{12}$. So we see that one-half of a period is $\dfrac{\pi}{2}$ and the period is π. So $B = \dfrac{2\pi}{\pi} = 2$.

4. For $y = A\cos(B(t - C)) + D$, we can use the point Q to determine a phase shift of $\dfrac{7\pi}{12}$. So an equation for this sinusoid is

$$y = 3\cos\left(2\left(t - \frac{7\pi}{12}\right)\right) + 4.$$

5. The point P is on the center line and so the horizontal distance between P and Q is one-quarter of a period. So this horizontal distance is $\dfrac{\pi}{4}$ and the t-coordinate of P is

$$\frac{7\pi}{12} - \frac{\pi}{4} = \frac{4\pi}{12} = \frac{\pi}{3}.$$

This can be the phase shift for $y = A\sin(B(t - C')) + D$. So another equation for this sinusoid is

$$y = 3\sin\left(2\left(t - \frac{\pi}{3}\right)\right) + 4.$$

Section 2.3

Progress Check 2.16

1. The maximum value of $V(t)$ is 140 ml and the minimum value of $V(t)$ is 70 ml. So the difference $(140 - 70 = 70)$ is twice the amplitude. Hence, the amplitude is 35 and we will use $A = 35$. The center line will then be 35 units below the maximum. That is, $D = 140 - 35 = 105$.

2. Since there are 50 beats per minute, the period is $\dfrac{1}{50}$ of a minute. Since we are using seconds for time, the period is $\dfrac{60}{50}$ seconds or $\dfrac{6}{5}$ sec. We can determine B by solving the equation

$$\frac{2\pi}{B} = \frac{6}{5}$$

for B. This gives $B = \dfrac{10\pi}{6} = \dfrac{5\pi}{3}$. Our function is

$$V(t) = 35\cos\left(\frac{5\pi}{3}t\right) + 105.$$

Progress Check 2.18

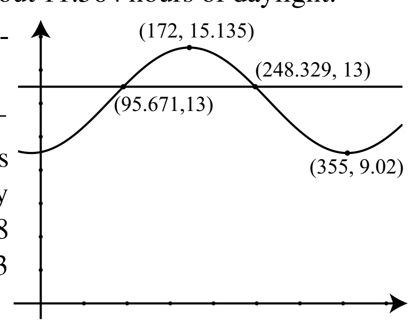
1. Since we have the coordinates for a high and low point, we first do the following computations:

   - $2(\text{amp}) = 15.35 - 9.02 = 6.33$. Hence, the amplitude is 3.165.
   - $D = 9.02 + 3.165 = 12.185$.
   - $\frac{1}{2}$period $= 355 - 172 = 183$. So the period is 366. Please note that we usually say that there are 365 days in a year. So it would also be reasonable to use a period of 365 days. Using a period of 366 days, we find that
   $$\frac{2\pi}{B} = 366,$$
   and hence $B = \dfrac{\pi}{183}$.

   We must now decide whether to use a sine function or a cosine function to get the phase shift. Since we have the coordinates of a high point, we will use a cosine function. For this, the phase shift will be 172. So our function is
   $$y = 3.165\cos\left(\frac{\pi}{183}(t - 172)\right) + 12.185.$$
   We can check this by verifying that when $t = 155$, $y = 15.135$ and that when $t = 355$, $y = 9.02$.

   (a) March 10 is day number 69. So we use $t = 69$ and get
   $$y = 3.165\cos\left(\frac{\pi}{183}(69 - 172)+\right) + 12.125 \approx 11.5642.$$
   So on March 10, 2014, there were about 11.564 hours of daylight.

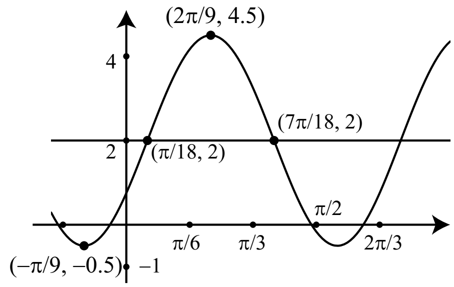
   (b) We use a graphing utility to approximate the points of intersection of $y = 3.165\cos\left(\frac{\pi}{183}(69 - 172)+\right) + 12.125$ and $y = 13$. The results are shown to the right. So on day 96 (April 6, 2014) and on day 248 (September 5), there were about 13 hours of daylight.



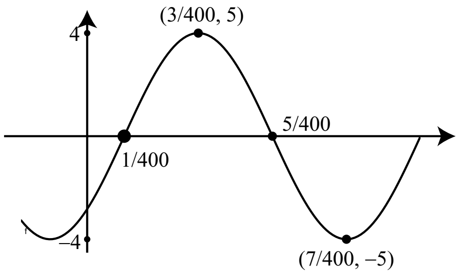
Progress Check 2.20

1.

(**a**) The amplitude is 2.5.
The period is $\dfrac{2\pi}{3}$.
The phase shift is $-\dfrac{\pi}{9}$.
The vertical shift is 2.



(**b**) The amplitude is 4.
The period is $\dfrac{1}{50}$.
The phase shift is $\dfrac{1}{400}$.
The vertical shift is 0.



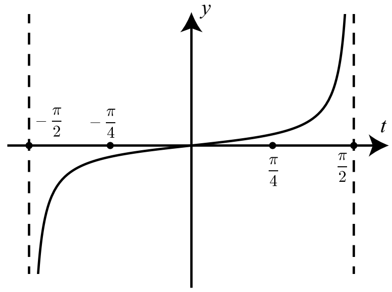
2.

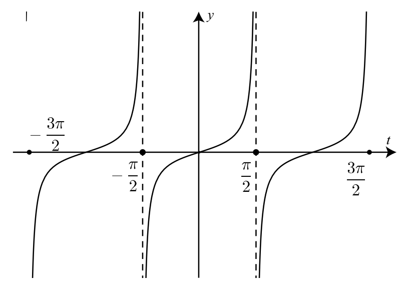
first equation		second equation
5.22	amplitude	5.153
12	period	12.30
3.7	phase shift	3.58
12.28	vertical shift	12.174

Section 2.4

Progress Check 2.21

The graphs for (1) and (2) are shown below.





3. In both graphs, the graph just to the left of $t = \dfrac{\pi}{2}$ and just to the right of $t = -\dfrac{\pi}{2}$ is consistent with the information in Table 2.4. The graph on the right is also consistent with the information in this table on both sides of $t = \dfrac{\pi}{2}$ and $t = -\dfrac{\pi}{2}$.

4. The range of the tangent function is the set of all real numbers.

5. Based on the graph in (2), the period of the tangent function appears to be π. The period is actually equal to π, and more information about this is given in Exercise (1).

Progress Check 2.23

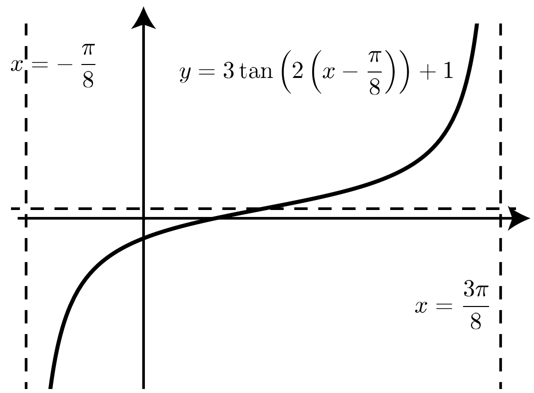
The equation for the function is $y = 3\tan\left(2\left(x - \dfrac{\pi}{8}\right)\right) + 1$.

1. The period of this function is $\dfrac{\pi}{2}$.

2. The effect of the parameter 3 is to vertically stretch the graph of the tangent function.

3. The effect of the parameter $\dfrac{\pi}{8}$ is to shift the graph of $y = 3\tan(2(x)) + 1$ to the right by $\dfrac{\pi}{8}$ units.

4. Following is a graph of one period of this function using $-\dfrac{\pi}{8} < x \leq \dfrac{3\pi}{8}$ and $-20 \leq y \leq 20$. The vertical asymptotes at $x = -\dfrac{\pi}{8}$ and $x = \dfrac{3\pi}{8}$ are shown as well as the horizontal line $y = 1$.



Progress Check 2.24

1. The secant function is the reciprocal of the cosine function. That is, $\sec(t) = \dfrac{1}{\cos(t)}$.

2. The domain of the secant function is the set of all real numbers t for which $t \neq \dfrac{\pi}{2} + k\pi$ for every integer k.

3. The graph of the secant function will have a vertical asymptote at those values of t that are not in the domain. So there will be a vertical asymptote when $t = \dfrac{\pi}{2} + k\pi$ for some integer k.

4. Since $\sec(t) = \dfrac{1}{\cos(t)}$, and the period of the cosine function is 2π, we conclude that the period of the secant function is also 2π.

Progress Check 2.26

1. All of the graphs are consistent.

2. Since $\sec(x) = \dfrac{1}{\cos(x)}$, we see that $\sec(x) > 0$ if and only if $\cos(x) > 0$. So the graph of $y = \sec(x)$ is above the x-axis if and only if the graph of $y = \cos(x)$ is above the x-axis.

3. Since $\sec(x) = \dfrac{1}{\cos(x)}$, we see that $\sec(x) < 0$ if and only if $\cos(x) < 0$. So the graph of $y = \sec(x)$ is below the x-axis if and only if the graph of $y = \cos(x)$ is below the x-axis.

4. The key is that $\sec(x) = \dfrac{1}{\cos(x)}$. Since $-1 \leq \cos(x) \leq 1$, we conclude that $\sec(x) \geq 1$ when $\cos(x) > 0$ and $\sec(x) \leq -1$ when $\cos(x) < 0$. Since the graph of the secant function has vertical asymptotes, we see that the range of the secant function consists of all real numbers y for which $y \geq 1$ or $y \leq -1$. This can also be seen on the graph of $y = \sec(x)$.

Section 2.5

Progress Check 2.28

1. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\pi}{3}$ since $\sin\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$ and $-\dfrac{\pi}{2} \leq -\dfrac{\pi}{3} \leq \dfrac{\pi}{2}$.

2. $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$ since $\sin\left(\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$ and $-\dfrac{\pi}{2} \leq \dfrac{\pi}{6} \leq \dfrac{\pi}{2}$.

3. $\arcsin(-1) = -\dfrac{\pi}{2}$ since $\sin\left(-\dfrac{\pi}{2}\right) = -1$ and $-\dfrac{\pi}{2} \leq -\dfrac{\pi}{2} \leq \dfrac{\pi}{2}$.

4. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$ since $\sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$ and $-\dfrac{\pi}{2} \leq -\dfrac{\pi}{4} \leq \dfrac{\pi}{2}$.

Progress Check 2.29

1. Since $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$, we see that $\sin\left(\sin^{-1}\left(\dfrac{1}{2}\right)\right) = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$.

2. $\arcsin\left(\sin\left(\dfrac{\pi}{4}\right)\right) = \arcsin\left(\dfrac{\sqrt{2}}{2}\right)$. In addition, $\arcsin\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ since $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and $-\dfrac{\pi}{2} \leq \dfrac{\pi}{4} \leq \dfrac{\pi}{2}$. So we see that

$$\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

3. We do not know an exact value for $\sin^{-1}\left(\dfrac{2}{5}\right)$. So we let $t = \sin^{-1}\left(\dfrac{2}{5}\right)$.

We then know that $\sin(t) = \dfrac{2}{5}$ and $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$. So

$$\sin\left(\sin^{-1}\left(\frac{2}{5}\right)\right) = \sin(t) = \frac{2}{5}.$$

4. $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right) = \arcsin\left(\dfrac{\sqrt{2}}{2}\right)$. In addition, $\arcsin\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ since $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and $-\dfrac{\pi}{2} \le \dfrac{\pi}{4} \le \dfrac{\pi}{2}$. So we see that

$$\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Progress Check 2.31

1. Since $\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$, we see that $\cos\left(\cos^{-1}\left(\dfrac{1}{2}\right)\right) = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$.

2. $\arccos\left(\cos\left(\dfrac{\pi}{4}\right)\right) = \arccos\left(\dfrac{\sqrt{2}}{2}\right)$. In addition, $\arccos\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ since $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and $0 \le \dfrac{\pi}{4} \le \pi$. So we see that

$$\arccos\left(\cos\left(\frac{\pi}{4}\right)\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

3. $\arccos\left(\cos\left(-\dfrac{\pi}{4}\right)\right) = \arccos\left(\dfrac{\sqrt{2}}{2}\right)$. In addition, $\arccos\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ since $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and $0 \le \dfrac{3\pi}{4} \le \pi$. So we see that

$$\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

4. $\tan^{-1}\left(\tan\left(\dfrac{5\pi}{4}\right)\right) = \tan^{-1}(1)$. In addition, $\tan^{-1}(1) = \dfrac{\pi}{4}$ since $\tan\left(\dfrac{\pi}{4}\right) = 1$ and $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$. So we see that

$$\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

Progress Check 2.32

1. $y = \arccos(1) = 0$
2. $y = \tan^{-1}\left(\sqrt{3}\right) = \dfrac{\pi}{3}$
3. $y = \arctan(-1) = -\dfrac{\pi}{4}$
4. $y = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\pi}{4}$
5. $\sin\left(\arccos\left(\dfrac{1}{2}\right)\right) = \dfrac{\sqrt{3}}{2}$
6. $\tan\left(\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)\right) = -\sqrt{3}$
7. $\arccos\left(\sin\left(\dfrac{\pi}{6}\right)\right) = \dfrac{\pi}{3}$

Progress Check 2.33

1. Let $t = \arccos\left(\dfrac{1}{3}\right)$. We then know that

$$\cos(t) = \frac{1}{3} \quad \text{and} \quad 0 \le t \le \pi.$$

Using the Pytagorean Identity, we see that $\left(\dfrac{1}{3}\right)^2 + \sin^2(t) = 1$ and this implies that $\sin^2(t) = \dfrac{8}{9}$. Since $0 \le t \le \pi$, t is in the second quadrant and in both of these quadrants, $\sin(t) > 0$. So, $\sin(t) = \frac{\sqrt{8}}{3}$. That is,

$$\sin\left(\arccos\left(\frac{1}{3}\right)\right) = \frac{\sqrt{8}}{3}.$$

2. For $\cos\left(\arcsin\left(-\dfrac{4}{7}\right)\right)$, we let $t = \arcsin\left(-\dfrac{4}{7}\right)$. This means that

$$\sin(t) = -\frac{4}{7} \quad \text{and} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

We can use the Pythagorean Identity to obtain $\cos^2(t) + \left(-\dfrac{4}{7}\right)^2 = 1$. This gives $\cos^2(t) = \dfrac{33}{49}$. We also have the restriction $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$ and we know $\sin(t) < 0$. This means that t must be in QIV and so $\cos(t) > 0$. Hence, $\cos(t) = \dfrac{\sqrt{33}}{7}$. That is,

$$\cos\left(\arcsin\left(-\frac{4}{7}\right)\right) = \frac{\sqrt{33}}{7}.$$

Note: You can use your calculator to check this work. Use your calculator to approximate both $\cos\left(\arcsin\left(-\frac{4}{7}\right)\right)$ and $\frac{\sqrt{33}}{7}$. Both results should be 0.8206518066.

Section 2.6

Progress Check 2.34

Any solution of the equation $\sin(x) = -0.6$ may be approximated with one of the following:

$$x \approx -0.64350 + k(2\pi) \quad \text{or} \quad x \approx -2.49809 + k(2\pi).$$

Progress Check 2.36

We first rewrite the equation $4\cos(x) + 3 = 2$ as follows:

$$\begin{aligned} 4\cos(x) + 3 &= 2 \\ 4\cos(x) &= -1 \\ \cos(x) &= -\frac{1}{4} \end{aligned}$$

So in the interval $[-\pi, \pi]$, the solutions are $x_1 = \arccos\left(-\frac{1}{4}\right)$ and $x_2 = -\arccos\left(-\frac{1}{4}\right)$. So any solution of the equation $4\cos(x) + 3 = 2$ is of the form

$$x = \arccos\left(-\frac{1}{4}\right) + k(2\pi) \quad \text{or} \quad x = -\arccos\left(-\frac{1}{4}\right) + k(2\pi).$$

Progress Check 2.38

We first use algebra to rewrite the equation $2\sin(x) + 1.2 = 2.5$ in the form

$$\sin(x) = 0.65.$$

So in the interval $[-\pi, \pi]$, the solutions are $x_1 = \arcsin(0.65)$ and $x_2 = \pi - \arcsin(0.65)$. So any solution of the equation $2\sin(x) + 1.2 = 2.5$ is of the form

$$x = \arcsin(0.65) + k(2\pi) \quad \text{or} \quad x = \pi - \arcsin(0.65) + k(2\pi).$$

Progress Check 2.39

1.

$$\begin{aligned} 3\cos(2x+1)+6 &= 5 \\ 3\cos(2x+1) &= -1 \\ \cos(2x+1) &= -\frac{1}{3} \end{aligned}$$

2. $t = \cos^{-1}\left(-\frac{1}{3}\right)$ or $t = -\cos^{-1}\left(-\frac{1}{3}\right)$.

3.

$$\begin{aligned} 2x+1 &= \cos^{-1}\left(-\frac{1}{3}\right) & \qquad 2x+1 &= -\cos^{-1}\left(-\frac{1}{3}\right) \\ 2x &= \cos^{-1}\left(-\frac{1}{3}\right)-1 & 2x &= -\cos^{-1}\left(-\frac{1}{3}\right)-1 \\ x &= \frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)-\frac{1}{2} & x &= -\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)-\frac{1}{2} \end{aligned}$$

4. The period of the function $y = \cos(2x+1)$ is π. So the following formulas can be used to generate the solutions for the equation.

$x = \left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)-\frac{1}{2}\right)+k\pi$ or $x = \left(-\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)-\frac{1}{2}\right)+k\pi$,

where k is some integer. Notice that we added an integer multiple of the period, which is π, to the solutions in (3).

Progress Check 2.40

We first write the equation $4\tan(x)+1 = 10$ in the form $\tan(x) = \frac{9}{4}$. So the only solution of the equation in the interval $\left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right)$ is

$$x = \arctan\left(\frac{9}{4}\right).$$

Since the period of the tangent function is π, any solution of this equation can be written in the form

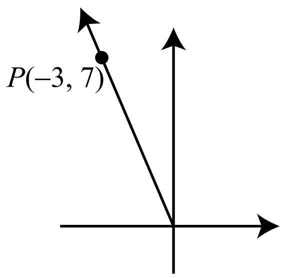
$$x = \arctan\left(\frac{9}{4}\right)+k\pi,$$

where k is some integer.

Section 3.1

Progress Check 3.1

1.



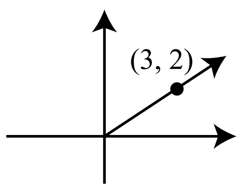
2. $r = \sqrt{(-3)^2 + 7^2} = \sqrt{58}$

3.

$$\cos(\theta) = -\frac{3}{\sqrt{58}} \qquad \tan(\theta) = -\frac{7}{3} \qquad \sec(\theta) = -\frac{\sqrt{58}}{3}$$
$$\sin(\theta) = \frac{7}{\sqrt{58}} \qquad \cot(\theta) = -\frac{3}{7} \qquad \csc(\theta) = \frac{\sqrt{58}}{7}$$

Progress Check 3.2

1.



2. Since $\tan(\alpha) = \dfrac{2}{3}$, we can conclude that the point $(3, 2)$ lies on the terminal side of α.

3. Since $(3, 2)$ is on the terminal side of α, we can use $x = 3$, $y = 2$, and $r = \sqrt{3^2 + 2^2} = \sqrt{13}$. So

$$\cos(\theta) = \frac{2}{\sqrt{13}} \qquad \tan(\theta) = \frac{2}{3} \qquad \sec(\theta) = \frac{\sqrt{13}}{2}$$
$$\sin(\theta) = \frac{3}{\sqrt{13}} \qquad \cot(\theta) = \frac{3}{2} \qquad \csc(\theta) = \frac{\sqrt{13}}{3}$$

Progress Check 3.3

The completed work should look something like the following:

$$\begin{aligned}\cos^2(\theta) + \sin^2(\theta) &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 \\ &= \frac{x^2}{r^2} + \frac{y^2}{r^2} \\ &= \frac{x^2 + y^2}{r^2} \\ &= \frac{r^2}{r^2} \\ &= 1\end{aligned}$$

Progress Check 3.4

1. Using the Pythagorean Identity, we see that $\cos^2(\theta) + \left(\frac{1}{3}\right)^2 = 1$ and so $\cos^2(\theta) = \frac{8}{9}$. Since $\frac{\pi}{2} < \theta < \pi$, $\cos(\theta) < 0$. Hence, $\cos(\theta) = -\frac{\sqrt{8}}{3}$.

2. $\tan(\theta) = \dfrac{\frac{1}{3}}{-\frac{\sqrt{8}}{3}} = -\dfrac{1}{\sqrt{8}}$.

3. $\cot(\theta) = -\sqrt{8}$, $\csc(\theta) = 3$, and $\sec(\theta) = -\frac{3}{\sqrt{8}}$.

Progress Check 3.5

2. $\tan^{-1}(-2.5) \approx -68.199°$.

3. $\theta \approx -68.199° + 180° \approx 111.801°$.

Section 3.2

Progress Check 3.9

We let α be the angle opposite the side of length 5 feet and let β be the angle adjacent to that side. We then see that

$$\sin(\alpha) = \frac{5}{17} \qquad\qquad \cos(\beta) = \frac{5}{17}$$
$$\alpha = \arcsin\left(\frac{5}{17}\right) \qquad\qquad \beta = \arccos\left(\frac{5}{17}\right)$$
$$\alpha \approx 17.1046^\circ \qquad\qquad \beta = 72.8954^\circ$$

As a check, we notice that $\alpha + \beta = 90^\circ$. We can use the Pythagorean theorem to determine the third side, which using our notation, is b. So

$$5^2 + b^2 = 17^2,$$

and so we see that $b = \sqrt{264} \approx 16.2481$ feet.

Progress Check 3.11

With a rise of 1 foot for every 12 feet of run, we see if we let θ be the angle of elevation, then

$$\tan(\theta) = \frac{1}{12}$$
$$\theta = \arctan\left(\frac{1}{12}\right)$$
$$\theta \approx 4.7636^\circ$$

The length of the ramp will be the hypotenuse of the right triangle. So if we let h be the length of the hypotenuse, then

$$\sin(\theta) = \frac{7.5}{h}$$
$$h = \frac{7.5}{\sin(\theta)}$$
$$h \approx 90.3120$$

The length of the hypotenuse is approximately 90.3 feet. We can check our result by determining the length of the third side, which is $7.5 \cdot 12$ or 90 feet and then verifying the result of the Pythagorean theorem. We can verify that

$$7.5^2 + 90^2 \approx 90.3120^2.$$

Progress Check 3.12

1. $h = x\tan(\alpha)$. So
$$\tan(\beta) = \frac{x\tan(\alpha)}{d + x}. \tag{3}$$

2. $\tan(\beta)(d + x) = x\tan(\alpha)$.

3. We can proceed to solve for x as follows:
$$\begin{aligned}
d\tan(\beta) + x\tan(\beta) &= x\tan(\alpha) \\
d\tan(\beta) &= x\tan(\alpha) - x\tan(\beta) \\
d\tan(\beta) &= x(\tan(\alpha) - \tan(\beta)) \\
\frac{d\tan(\beta)}{\tan(\alpha) - \tan(\beta)} &= x
\end{aligned}$$
So we see that $x = \dfrac{22.75\tan(34.7°)}{\tan(43.2°) - \tan(34.7°)} \approx 63.872$. Using this value for x, we obtain $h = x\tan(43.2°) \approx 59.980$. So the top of the flagpole is about 59.98 feet above the ground.

4. There are several ways to check this result. One is to use the values for d, h, and x and the inverse tangent function to determine the values for α and β. If we use approximate values for d, h, and x, these checks may not be exact. For example,
$$\begin{aligned}
\alpha &= \arctan\left(\frac{h}{x}\right) \approx \arctan\left(\frac{59.98}{63.872}\right) \approx 43.2° \\
\beta &= \arctan\left(\frac{h}{d + x}\right) \approx \arctan\left(\frac{59.980}{22.75 + 63.872}\right) \approx 34.7°
\end{aligned}$$
Another method to check the results is to use the sine of α or β to determine the length of the hypotenuse of one of the right triangles and then check using the Pythagorean Theorem.

Section 3.3

Progress Check 3.14

We first note that the third angle in the triangle is 30° since the sum of the two given angles is 150°. We let x be the length of the side opposite the 15° angle and let y be the length of the side opposite the 135° angle. We then see that

$$\begin{aligned} \frac{x}{\sin(15°)} &= \frac{71}{\sin(30°)} & \frac{y}{\sin(135°)} &= \frac{71}{\sin(30°)} \\ x &= \frac{71\sin(15°)}{\sin(30°)} & y &= \frac{71\sin(135°)}{\sin(30°)} \\ x &\approx 36.752 & y &\approx 100.409 \end{aligned}$$

So the length of the side opposite the 15° angle is about 36.75 inches, and the length of the side opposite the 135° angle is about 100.41 inches.

Progress Check 3.15

1. The side opposite the angle of 40° has length 1.7 feet. So we get

$$\begin{aligned} \frac{\sin(\theta)}{2} &= \frac{\sin(40°)}{1.7} \\ \sin(\theta) &= \frac{2\sin(40°)}{1.7} \approx 0.75622 \end{aligned}$$

2. We see that

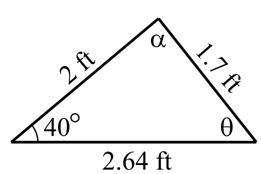
$$\theta_1 = \sin^{-1}\left(\frac{2\sin(40°)}{1.7}\right) \approx 49.132°.$$

3. $\theta_2 = 180° - \theta_1 \approx 130.868°$. Using reference angles instead of reference arcs, θ_1 is the reference angle for θ_2, which is in the second quadrant. Hence, $\sin(\theta_2) = \sin(\theta_1)$.

4. The third angle α can be determined using the sum of the angles of a triangle.

$$\begin{aligned} \alpha + \theta_1 + 40° &= 180° \\ \alpha &\approx 180° - 40° - 49.132° \\ \alpha &\approx 90.868° \end{aligned}$$

We use the Law of Sines to determine the length x of the side opposite α. The resulting triangle is shown on the right.

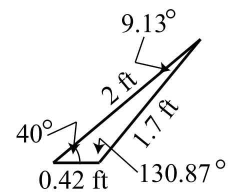


$$\begin{aligned} \frac{x}{\sin(\alpha)} &= \frac{1.7}{\sin(40°)} \\ x &= \frac{1.7\sin(\alpha)}{\sin(40°)} \\ x &\approx 2.644 \text{ ft} \end{aligned}$$

5. Using the same procedure that we did in part (4), we obtain

$$\begin{aligned}\theta_2 &\approx 130.868°\\ \alpha_2 &\approx 9.132°\\ x_2 &\approx 0.420 \text{ ft}\end{aligned}$$

The triangle is shown on the right.



Progress Check 3.16

1. The side opposite the angle of 40° has length 3 feet. So we get

$$\begin{aligned}\frac{\sin(\theta)}{2} &= \frac{\sin(40°)}{3}\\ \sin(\theta) &= \frac{2\sin(40°)}{3} \approx 0.42853\end{aligned}$$

2. We see that

$$\theta_1 = \sin^{-1}\left(\frac{2\sin(40°)}{3}\right) \approx 25.374°.$$

3. $\theta_2 = 180° - \theta_1 \approx 154.626°$. Using reference angles instead of reference arcs, θ_1 is the reference angle for θ_2, which is in the second quadrant. Hence, $\sin(\theta_2) = \sin(\theta_1)$.

4. The third angle α can be determined using the sum of the angles of a triangle.

$$\begin{aligned}\alpha + \theta_1 + 40° &= 180°\\ \alpha &\approx 180° - 40° - 25.374°\\ \alpha &\approx 114.626°\end{aligned}$$

We use the Law of Sines to determine the length x of the side opposite α. The resulting triangle is shown on the right.

$$\begin{aligned}\frac{x}{\sin(\alpha)} &= \frac{3}{\sin(40°)}\\ x &= \frac{3\sin(\alpha)}{\sin(40°)}\\ x &\approx 4.243 \text{ ft}\end{aligned}$$

5. Using the same procedure that we did in part (4), we obtain

$$\begin{aligned}\theta_2 &\approx 154.626^\circ \\ 40^\circ + \theta_2 &= 194.626\circ\end{aligned}$$

This is not possible since the sum of the angles of a triangle is 180°. So there is no triangle where the angle the angle opposite the side of length 2 is θ_2.

Progress Check 3.17

1. Using the Law of Cosines, we obtain

$$\begin{aligned}c^2 &= 3.5^2 + 2.5^2 - 2(3.5)(2.5)\cos\left(60^\circ\right) \\ &= 9.75\end{aligned}$$

So $c = \sqrt{9.75} \approx 3.12250$ ft.

2. Using the Law of Sines, we obtain

$$\begin{aligned}\frac{\sin(\alpha)}{2.5} &= \frac{\sin\left(60^\circ\right)}{c} \\ \sin(\alpha) &= \frac{2.5\sin\left(60^\circ\right)}{c} \approx 0.69338\end{aligned}$$

From this, we get $\alpha \approx 43.898^\circ$ or $\alpha \approx 136.102^\circ$. However, since the given angle in 60°, the second value is not possible since $136.102^\circ + 60^\circ < 180^\circ$. So $\alpha \approx 43.898^\circ$.

3. Since the sum of the angles of a triangle must be 180°, we have

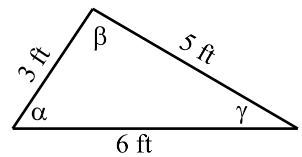
$$\begin{aligned}60^\circ + 43.898^\circ + \beta &= 180^\circ \\ \beta &\approx 76.102^\circ\end{aligned}$$

4. With the values we have determined, we can check our work by showing that

$$\frac{\sin\left(60^\circ\right)}{c} = \frac{\sin(\alpha)}{2.5} = \frac{\sin(\beta)}{3.5} \approx 0.27735.$$

Progress Check 3.18

The first step is to draw a reasonably accurate diagram and label the angles. We will use the diagram on the right. Using the Law of Cosines, we obtain



$$
\begin{aligned}
5^2 &= 3^2 + 6^2 - 2(3)(6)\cos(\alpha) \\
\cos(\alpha) &= \frac{20}{36} \\
\alpha &\approx 56.251^\circ
\end{aligned}
\qquad
\begin{aligned}
6^2 &= 3^2 + 5^2 - 2(3)(5)\cos(\beta) \\
\cos(\beta) &= \frac{-2}{30} \\
\beta &\approx 98.823^\circ
\end{aligned}
$$

$$
\begin{aligned}
3^2 &= 5^2 + 6^2 - 2(5)(6)\cos(\gamma) \\
\cos(\gamma) &= \frac{52}{60} \\
\gamma &\approx 29.926^\circ
\end{aligned}
$$

We check these results by verifying that $\alpha + \beta + \gamma = 180^\circ$.

Section 3.4

Progress Check 3.20

We first note that $\angle BAC = 180^\circ - 94.2^\circ - 48.5^\circ$ and so $\angle BAC = 37.3^\circ$. We can then use the Law of Sines to determine the length from A to B as follows:

$$
\begin{aligned}
\frac{AB}{\sin(48.5^\circ)} &= \frac{98.5}{\sin(37.3^\circ)} \\
AB &= \frac{98.5\sin(48.5^\circ)}{\sin(37.3^\circ)} \\
AB &\approx 121.7
\end{aligned}
$$

The bridge from point B to point A will be approximately 121.7 feet long.

Progress Check 3.21

Using the right triangle, we see that $\sin(26.5°) = \dfrac{h}{5}$. So $h = 5\sin(26.5°)$, and the area of the triangle is

$$\begin{aligned} A &= \frac{1}{2}(7)\left[5\sin\left(26.5°\right)\right] \\ &= \frac{35}{2}\sin\left(26.5°\right) \approx 7.8085 \end{aligned}$$

The area of the triangle is approximately 7.8085 square meters.

Progress Check 3.22

Using the right triangle, we see that $\sin(\theta) = \dfrac{h}{a}$. So $h = a\sin(\theta)$, and the area of the triangle is

$$\begin{aligned} A &= \frac{1}{2}b\left(a\sin(\theta)\right) \\ &= \frac{1}{2}ab\sin(\theta) \end{aligned}$$

Progress Check 3.23

1. Using the Law of Cosines, we see that

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\gamma) \\ 2ab\cos(\gamma) &= a^2 + b^2 - c^2 \\ \cos(\gamma) &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

2. We see that

$$\sin^2(\gamma) = 1 - \cos^2(\gamma).$$

Since γ is between 0° and 180°, we know that $\sin(\gamma) > 0$ and so

$$\sin(\gamma) = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}.$$

3. Substituting the equation in part (2) into the formula $A = \frac{1}{2}ab\sin(\gamma)$, we obtain

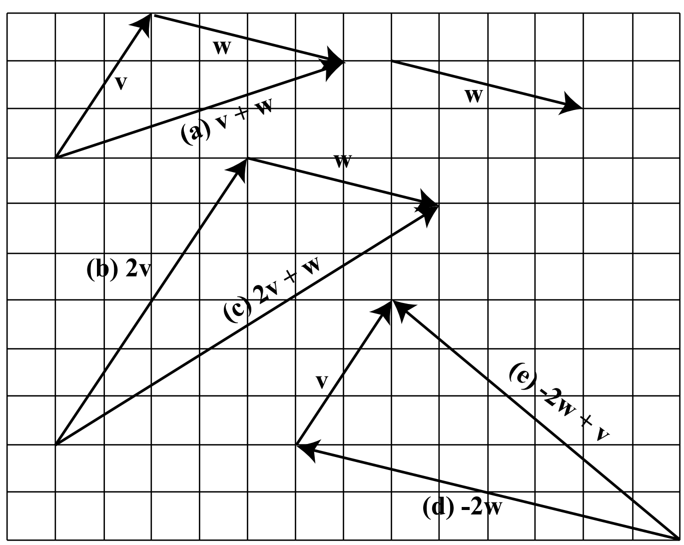
$$\begin{aligned} A &= \frac{1}{2}ab\sin(\gamma) \\ &= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \end{aligned}$$

Section 3.5

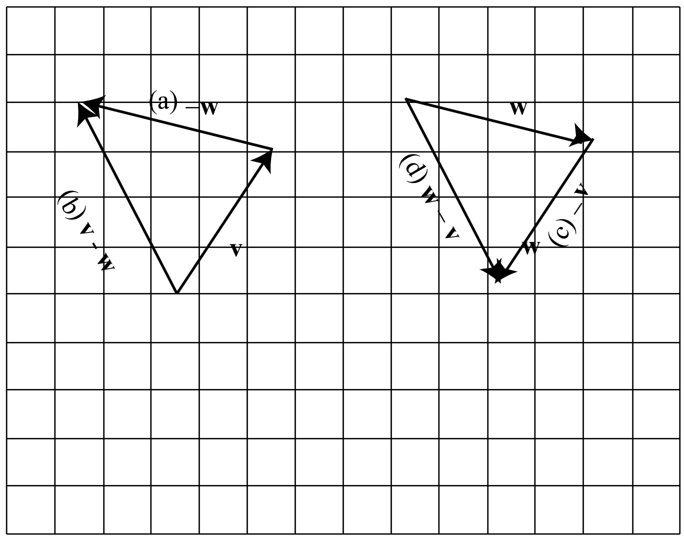
Progress Check 3.24

The vector **w** is the only vector that is equal to the vector **v**. Vector **u** has the same direction as **v** but a different magnitude. Vector **a** has the same magnitude as **v** but a different direction (note that the direction of **a** is the opposite direction of **v**). Vector **b** has a different direction and a different magnitude than **v**.

Progress Check 3.25



Progress Check 3.26



Progress Check 3.27

1. $\angle ABC = 180° - \theta = 127°$.

2. Using the Law of Cosines, we see that

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\cdot|\mathbf{b}|\cos(\angle ABC) \\ &= 80^2 + 60^2 - 2\cdot 60\cdot 80\cos\left(127°\right) \\ &= 10000 - 9600\cos\left(127°\right) \\ &\approx 15777.42422 \end{aligned}$$

So we see that $|\mathbf{a}+\mathbf{b}| \approx 125.61$.

3. The angle between the vectors **a** and $\mathbf{a}+\mathbf{b}$ is $\angle CAB$. In $\triangle ABC$, we know that $\angle ABC = 127°$, and so $\angle CAB$ must be an acute angle. We will use the Law of Sines to determine this angle.

$$\begin{aligned} \frac{\sin(\angle CAB)}{|\mathbf{b}|} &= \frac{\sin(\angle ABC)}{|\mathbf{a}+\mathbf{b}|} \\ \sin(\angle CAB) &= \frac{60\sin\left(127°\right)}{|\mathbf{a}+\mathbf{b}|} \\ \sin(\angle CAB) &\approx 0.38148341 \end{aligned}$$

So the angle between the vectors **a** and $\mathbf{a}+\mathbf{b}$ is approximately $22.43°$.

Progress Check 3.29

Using the Law of Sines, we see that

$$\frac{|\mathbf{a}|}{\sin(20^\circ)} = \frac{100}{\sin(140^\circ)}$$
$$|\mathbf{a}| = \frac{100\sin(20^\circ)}{\sin(140^\circ)}$$
$$|\mathbf{a}| \approx 53.21$$

The magnitude of the vector **a** (and the vector **b**) is approximately 53.21 pounds.

Progress Check 3.30

Using the notation in Figure 3.28, we obtain the following:

$$\frac{|\mathbf{b}|}{|\mathbf{w}|} = \cos\left(12^\circ\right) \qquad \frac{|\mathbf{a}|}{|\mathbf{w}|} = \sin\left(12^\circ\right)$$
$$|\mathbf{b}| = |\mathbf{w}|\cos\left(12^\circ\right) \qquad |\mathbf{a}| = |\mathbf{w}|\sin\left(12^\circ\right)$$
$$|\mathbf{b}| \approx 244.54 \qquad |\mathbf{a}| \approx 51.98$$

The object exerts a force of about 244.54 pounds perpendicular to the plane and the force of gravity down the plane on the object is about 51.98 pounds. So in order to keep the object stationary, a force of about 51.98 pounds up the plane must be applied to the object.

Section 3.6

Progress Check 3.31

1. $\mathbf{v} = 7\mathbf{i} + (-3)\mathbf{j}$. So $|\mathbf{v}| = \sqrt{7^2 + (-3)^2} = \sqrt{58}$. In addition,

$$\cos(\theta) = \frac{7}{\sqrt{58}} \quad \text{and} \quad \sin(\theta) = \frac{-3}{\sqrt{58}}.$$

So the terminal side of θ is in the fourth quadrant, and we can write

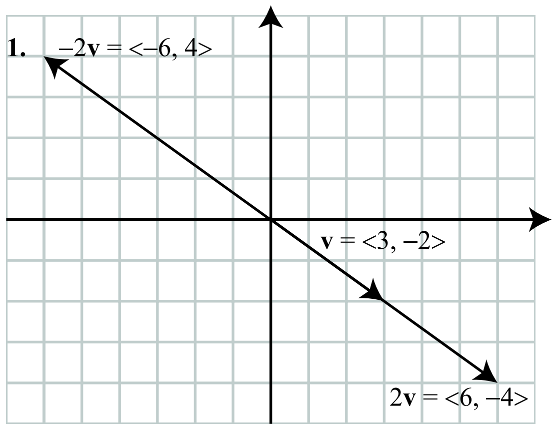
$$\theta = 360^\circ - \arccos\left(\frac{7}{\sqrt{58}}\right).$$

So $\theta \approx 336.80^\circ$.

2. We are given $|\mathbf{w}| = 20$ and the direction angle θ of $\mathbf{w}$ is $200°$. So if we write $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$, then

$$w_1 = 20\cos\left(200°\right) \approx -1.794 \qquad w_2 = 20\sin\left(200°\right) \approx -6.840$$

Progress Check 3.32



2. For a vector $\mathbf{a} = \langle a_1, a_2\rangle$ and a scalar c, we define the scalar multiple $c\mathbf{a}$ to be

$$c\mathbf{a} = \langle ca_1, ca_2\rangle.$$

Progress Check 3.33

Let $\mathbf{u} = \langle 1, -2\rangle$, $\mathbf{v} = \langle 0, 4\rangle$, and $\mathbf{w} = \langle -5, 7\rangle$.

1. $2\mathbf{u} - 3\mathbf{v} = \langle 2, -4\rangle - \langle 0, 12\rangle = \langle 2, -16\rangle$.

2. $|2\mathbf{u} - 3\mathbf{v}| = \sqrt{2^2 + (-16)^2} = \sqrt{260}$. So now let θ be the direction angle of $2\mathbf{u} - 3\mathbf{v}$. Then

$$\cos(\theta) = \frac{2}{\sqrt{260}} \quad \text{and} \quad \sin(\theta) = \frac{-16}{\sqrt{260}}.$$

So the terminal side of θ is in the fourth quadrant. We see that $\arcsin\left(\dfrac{-16}{\sqrt{260}}\right) \approx -82.87°$. Since the direction angle θ must satisfy $0 \le \theta < 360°$, we see that $\theta \approx -82.87° + 360° \approx 277.13°$.

3. $\mathbf{u} + 2\mathbf{v} - 7\mathbf{w} = \langle 1, -2\rangle + \langle 0, 8\rangle - \langle -35, 49\rangle = \langle 36, -43\rangle$.

Progress Check 3.34

1. If θ is the angle between $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$, then

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-13}{\sqrt{10}\sqrt{29}}$$
$$\theta = \cos^{-1}\left(\frac{-13}{\sqrt{10}\sqrt{29}}\right)$$

so $\theta \approx 139.764°$.

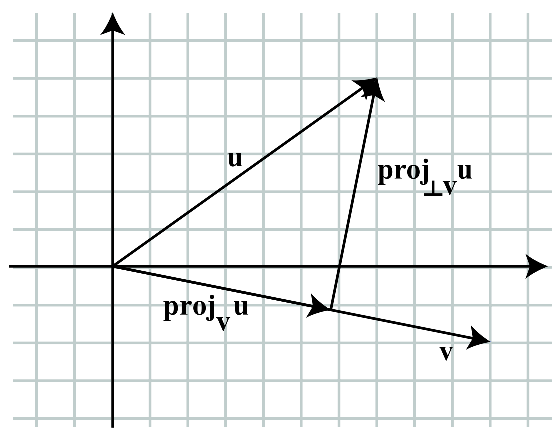
2. If $\mathbf{v} = \langle a, b \rangle$ is perpendicular to $\mathbf{u} = \langle 1, 3 \rangle$, then the angle θ between them is $90°$ and so $\cos(\theta) = 0$. So we must have $\mathbf{u}\cdot\mathbf{v} = 0$ and this means that $a + 3b = 0$. So any vector $\mathbf{v} = \langle a, b \rangle$ where $a = -3b$ will be perpendicular to $\mathbf{v}$, and there are infinitely many such vectors. One vector perpendicular to $\mathbf{u}$ is $\langle -3, 1 \rangle$.

Progress Check 3.35

Let $\mathbf{u} = \langle 7, 5 \rangle$ and $\mathbf{v} = \langle 10, -2 \rangle$. Then

$$\begin{aligned}\mathbf{proj_v u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{60}{104}\mathbf{v} \\ &= \left\langle \frac{600}{104}, \frac{-120}{104} \right\rangle \\ &\approx \langle 5.769, -1.154 \rangle\end{aligned}$$

$$\begin{aligned}\mathbf{proj_{\perp v} u} &= \mathbf{u} - \mathbf{proj_v u} \\ &= \langle 7, 5 \rangle - \left\langle \frac{600}{104}, \frac{-120}{104} \right\rangle \\ &= \left\langle \frac{128}{104}, \frac{640}{104} \right\rangle \\ &\approx \langle 1.231, 6.154 \rangle\end{aligned}$$



Section 4.1

Progress Check 4.4

1. The graphs of both sides of the equation indicate that this is an indentity.
2. The graphs of both sides of the equation indicate that this is not an indentity. For example, if we let $x = \dfrac{\pi}{2}$, then

$$\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 0 \cdot 1 = 0 \qquad \text{and} \qquad 2\sin\left(\frac{\pi}{2}\right) = 2 \cdot 1 = 2.$$

.

Section 4.2

Progress Check 4.6

We divide both sides of the equation $4\cos(x) = 2\sqrt{2}$ by 4 to get $\cos(x) = \dfrac{\sqrt{2}}{2}$. So

$$x = \frac{\pi}{4} + k(2\pi) \quad \text{or} \quad x = \frac{7\pi}{4} + k(2\pi),$$

where k is an integer.

Progress Check 4.7

1. We divide both sides of the equation $5\sin(x) = 2$ by 5 to get $\sin(x) = 0.4$. So

$$x = \sin^{-1}(0.4) + k(2\pi) \quad \text{or} \quad x = (\pi - \sin^{-1}(0.4)) + k(2\pi),$$

where k is an integer.

2. We use $\alpha = 40°$ and $\dfrac{c_a}{c_w} = 1.33$ in the Law of Refraction.

$$\frac{\sin(40°)}{\sin(\beta)} = 1.33$$
$$\sin(\beta) = \frac{\sin(40°)}{1.33} \approx 0.483299$$
$$\beta \approx 28.90°$$

The angle of refraction is approximately $28.90°$.

Progress Check 4.9

We will use the identity $\cos^2(x) = 1 - \sin^(x)$. So we have

$$\begin{aligned}\sin^2(x) &= 3\left(1 - \sin^2(x)\right)\\ \sin^2(x) &= \frac{3}{4}\end{aligned}$$

So we have $\sin(x) = \dfrac{\sqrt{3}}{2}$ or $\sin(x) = -\dfrac{\sqrt{3}}{2}$. For the first equation, we see that

$$x = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad x = \frac{\pi}{3} + 2\pi k,$$

where k is an integer, and for the second equation, we have

$$x = \frac{4\pi}{3} + 2\pi k \quad \text{or} \quad x = \frac{5\pi}{3} + 2\pi k,$$

where k is an integer. The graphs of $y = \sin^2(x)$ and $y = 3\cos^2(x)$ will show 4 points of intersection on the interval $[0, 2\pi]$.

Progress Check 4.11

We write the equation as $\sin^2(x) - 4\sin(x) + 3 = 0$ and factor the right side to get $(\sin(x) - 3)(\sin(x) - 1) = 0$. So we see that $\sin(x) - 3 = 0$ or $\sin(x) - 1 = 0$. However, the equation $\sin(x) - 3 = 0$ is equivalent to $\sin(x) = 3$, and this equation has no solution. We write $\sin(x) - 1 = 0$ as $\sin(x) = 1$ and so the solutions are

$$x = \frac{\pi}{2} + 2\pi k,$$

where k is an integer.

Section 4.3

Progress Check 4.13

1. We first note that $\dfrac{7\pi}{12} = \dfrac{9\pi}{12} - \dfrac{2\pi}{6} = \dfrac{3\pi}{4} - \dfrac{\pi}{6}$.

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{-\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

2.

$$\begin{aligned}\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{6} - \left(-\frac{\pi}{4}\right)\right)\\ &= \cos\left(\frac{\pi}{6}\right)\cos\left(-\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{4}\right)\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

Progress Check 4.14

1. $\cos(\pi + x) = \cos(\pi)\cos(x) - \sin(\pi)(\sin(x) = -\cos(x)$. The graphs of $y = \cos(\pi + x)$ and $y = \cos(x)$ are indentical.

2. $\cos\left(\dfrac{\pi}{2} - x\right) = \cos\left(\dfrac{\pi}{2}\right)\cos(x) + \sin(\left(\dfrac{\pi}{2}\right)\sin(x) = 0 \cdot \cos(x) + 1 \cdot \sin(x)$.

   So we see that $\cos\left(\dfrac{\pi}{2} - x\right) = \sin(x)$

Progress Check 4.15

We will use the identity $\tan(y) = \dfrac{\sin(y)}{\cos(y)}$.

$$\begin{aligned}\tan\left(\frac{\pi}{2} - x\right) &= \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} \\ &= \frac{\cos(x)}{\sin(x)} \\ &= \cot(x)\end{aligned}$$

Progress Check 4.16

1. We note that $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$.

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

2. We note that $\dfrac{5\pi}{12} = \dfrac{\pi}{4} + \dfrac{\pi}{6}$.

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

Progress Check 4.18

We first use the Sine Sum Identity to rewrite the equation as $\sin(x + 1) = 0.2$. If we let $t = x + 1$, we see that for $0 \le t < 2\pi$,

$$t = \arcsin(0.2) \quad \text{or} \quad t = (\pi - \arcsin(0.2)).$$

So we have $x + 1 = \arcsin(0.2)$ or $x + 1 = \pi - \arcsin(0.2)$. Since the period of the functions we are working with is 2π, we see that

$$x = (-1 + \arcsin(0.2)) + k(2\pi) \text{ or } x = (-1 + \pi - \arcsin(0.2)) + k(2\pi),$$

where k is an integer.

Section 4.4

Progress Check 4.19

We are assuming that $\cos(\theta) = \dfrac{5}{13}$ and $\dfrac{3\pi}{2} \leq \theta \leq 2\pi$. To determine $\cos(2\theta)$ and $\sin(2\theta)$, we will use the double angle indentities.

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad \sin(2\theta) = 2\cos(\theta)\sin(\theta).$$

To use these identities, we also need to know $\sin(\theta)$. So we use the Pythagorean identity.

$$\begin{aligned}
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
\sin^2(\theta) &= 1 - \cos^2(\theta) \\
&= 1 - \left(\frac{5}{13}\right)^2 \\
&= \frac{144}{169}
\end{aligned}$$

Since $\dfrac{3\pi}{2} \leq \theta \leq 2\pi$, we see that $\sin(\theta) < 0$ and so $\sin(\theta) = -\dfrac{12}{13}$. Hence,

$$\begin{aligned}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) & \sin(2\theta) &= 2\cos(\theta)\sin(\theta) \\
&= \left(\frac{5}{13}\right)^2 - \left(-\frac{12}{13}\right)^2 & &= 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) \\
&= -\frac{119}{169} & &= -\frac{120}{169}
\end{aligned}$$

Progress Check 4.20

We will prove alternate forms of the double angle identity for cosine.

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= \left(1 - \sin^2(A)\right) - \sin^2(A) \\ &= 1 - \sin^2(A) - \sin^2(A) \\ &= 1 - 2\sin^2(A)\end{aligned}$$

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= \cos^2(A) - \left(1 - \cos^2(A)\right) \\ &= \cos^2(A) - 1 + \cos^2(A) \\ &= 2\cos^2(A) - 1\end{aligned}$$

Progress Check 4.22

We will approximate the smallest positive solution in degrees, to two decimal places, to the range equation

$$45000 \sin(2\theta) = 1000.$$

Dividing both sides of the equation by 45000, we obtain

$$\sin(2\theta) = \frac{1000}{45000} = \frac{1}{45}.$$

So

$$\begin{aligned}2\theta &= \arcsin\left(\frac{1}{45}\right) \\ \theta &= \frac{1}{2}\arcsin\left(\frac{1}{45}\right)\end{aligned}$$

Using a calculator in degree mode, we obtain $\theta \approx 0.64°$.

Progress Check 4.24

1. We use the double angle identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ to obtain

$$\begin{aligned}1 - 2\sin^2(\theta) &= \sin(\theta) \\ 1 - 2\sin^2(\theta) - \sin(\theta) &= 0 \\ 2\sin^2(\theta) + \sin(\theta) - 1 &= 0\end{aligned}$$

2. Factoring gives $(2\sin(\theta) - 1)(\sin(\theta) + 1) = 0$. Setting each factor equal to 0 and solving for $\sin(\theta)$, we obtain

$$\sin(\theta) = \frac{1}{2} \quad \text{or} \quad \sin(\theta) = -1.$$

So we have

$$\theta = \frac{\pi}{6} + k(2\pi) \text{ or } \theta = \frac{5\pi}{6} + k(2\pi) \text{ or } \theta = -\frac{pi}{2} + k(2pi),$$

where k is an integer.

Progress Check 4.26

To determine the exact value of $\cos\left(\frac{\pi}{8}\right)$, we use the Half Angle Identity for cosine with $A = \frac{\pi}{4}$.

$$\begin{aligned}\cos\left(\frac{\pi}{8}\right) &= \pm\sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} \\ &= \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}\end{aligned}$$

Since $\frac{\pi}{8}$ is in the first quadrant, we will use the positive square root. We can also rewrite the expression under the square root sign to obtain

$$\begin{aligned}\cos\left(\frac{\pi}{8}\right) &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2+\sqrt{2}}{4}} \\ &= \frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

This result can be checked using a calculator.

Section 4.5

Progress Check 4.27

To determine the exact value of $\sin(52.5°)\sin(7.5°)$, we will use the Product-to-Sum identity

$$\sin(A)\sin(B) = \left(\frac{1}{2}\right)[\cos(A-B) - \cos(A+B)].$$

So we see that

$$\begin{aligned}\sin(52.5°)\sin(7.5°) &= \left(\frac{1}{2}\right)\left[\cos(45°) - \cos(60°)\right] \\ &= \left(\frac{1}{2}\right)\left[\frac{\sqrt{2}}{2} - \frac{1}{2}\right] \\ &= \frac{\sqrt{2}-1}{4}\end{aligned}$$

Progress Check 4.28

To determine the exact value of $\cos(112.5°) + \cos(67.5°)$, we will use the Sum-to-Product Identity

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

So we see that

$$\begin{aligned}\cos(112.5°) + \cos(67.5°) &= 2\cos\left(\frac{180°}{2}\right)\cos\left(\frac{45°}{2}\right) \\ &= \cos(90°)\cos(22.5°) \\ &= 0 \cdot \cos(22.5°) \\ &= 0\end{aligned}$$

Section 5.1

Progress Check 5.1

1. **(a)** $(2+3i) + (7-4i) = 9 - i$

(b) $(4-2i)(3+i) = (4-2i)3 + (4-2i)i = 14 - 2i$

(c) $(2+i)i - (3+4i) = (2i-1) - 3 - 4i = -4 - 2i$

2. We use the quadratice formula to solve the equation and obtain $x = \dfrac{1 \pm \sqrt{-7}}{2}$. We can then write $\sqrt{-7} = i\sqrt{7}$. So the two solutions of the quadratic equation are:

$$x = \frac{1 + i\sqrt{7}}{2} \qquad\qquad x = \frac{1 - i\sqrt{7}}{2}$$

$$x = \frac{1}{2} + \frac{\sqrt{7}}{2}i \qquad\qquad x = \frac{1}{2} - \frac{\sqrt{7}}{2}i$$

Progress Check 5.5

1. Using our formula with $a = 5$, $b = -1$, $c = 3$, and $d = 4$ gives us

$$\frac{5 - i}{3 + 4i} = \frac{15 - 4}{15} + \frac{-3 - 20}{25}i = \frac{11}{25} - \frac{23}{25}i.$$

As a check, we see that

$$\begin{aligned}\left(\frac{11}{25} - \frac{23}{25}i\right)(3 + 4i) &= \left(\frac{33}{25} - \frac{69}{25}i\right) + \frac{44}{25}i - \frac{92}{25}i^2 \\ &= \left(\frac{33}{25} + \frac{92}{25}\right) + \left(-\frac{69}{25}i + \frac{44}{25}i\right) \\ &= 5 - i\end{aligned}$$

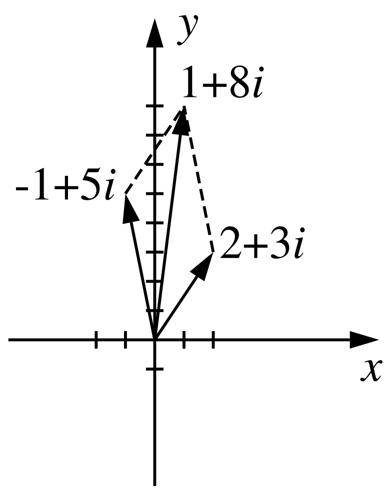
2. We can solve for x by dividing both sides of the equation by $3 + 4i$ to see that

$$x = \frac{5 - i}{3 + 4i} = \frac{11}{25} - \frac{23}{25}i.$$

Progress Check 5.2

1. The sum is $w + z = (2 - 1) + (3 + 5)i = 1 + 8i$.

2. A representation of the complex sum using vectors is shown in the figure below.



Progress Check 5.3

1. Using the definition of the conjugate of a complex number we find that $\overline{w} = 2 - 3i$ and $\overline{z} = -1 - 5i$.

2. Using the definition of the norm of a complex number we find that $|w| = \sqrt{2^2 + 3^3} = \sqrt{13}$ and $|z| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}$.

3. Using the definition of the product of complex numbers we find that

$$\begin{aligned} w\overline{w} &= (2 + 3i)(2 - 3i) = 4 + 9 = 13 \\ z\overline{z} &= (-1 + 5i)(-1 - 5i) = 1 + 25 = 26. \end{aligned}$$

4. Let $z = a + 0i = a$ for some $a \in \mathbb{R}$. Then $\overline{z} = a - 0i = a$. Thus, $\overline{z} = z$ when $z \in \mathbb{R}$.

Section 5.2

Progress Check 5.6

1. Note that $|w| = \sqrt{4^2 + (4\sqrt{3})^2} = 4\sqrt{4} = 8$ and the argument of w is $\arctan\left(\frac{4\sqrt{3}}{4}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$. So

$$w = 8\left(\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right).$$

Also, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and the argument of z is $\arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$. So

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\right)$$
$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right).$$

2. Recall that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. So

$$3\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i.$$

So $a = \frac{3\sqrt{3}}{2}$ and $b = \frac{3}{2}$.

Progress Check 5.8

1. Since $|w| = 3$ and $|z| = 2$, we see that

$$|wz| = |w||z| = (3)(2) = 6.$$

2. The argument of w is $\dfrac{5\pi}{3}$ and the argument of z is $-\dfrac{\pi}{4}$, we see that the argument of wz is

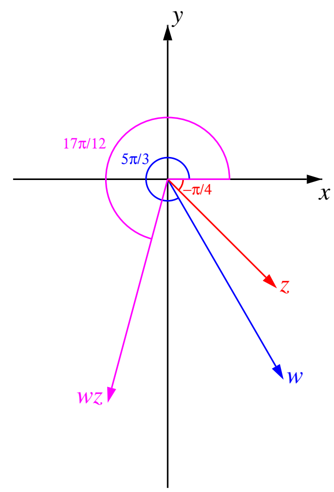
$$\frac{5\pi}{3} - \frac{\pi}{4} = \frac{20\pi - 3\pi}{12} = \frac{17\pi}{12}.$$

3. The terminal side of an angle of $\dfrac{17\pi}{12} = \pi + \dfrac{5\pi}{12}$ radians is in the third quadrant.

4. We know the magnitude and argument of wz, so the polar form of wz is

$$wz = 6\left[\cos\left(\frac{17\pi}{12}\right) + \sin\left(\frac{17\pi}{12}\right)\right].$$

5. Following is a picture of w, z, and wz that illustrates the action of the complex product.



Progress Check 5.9

1. Since $|w| = 3$ and $|z| = 2$, we see that

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{3}{2}.$$

2. The argument of w is $\dfrac{5\pi}{3}$ and the argument of z is $-\dfrac{\pi}{4}$, we see that the argument of $\dfrac{w}{z}$ is

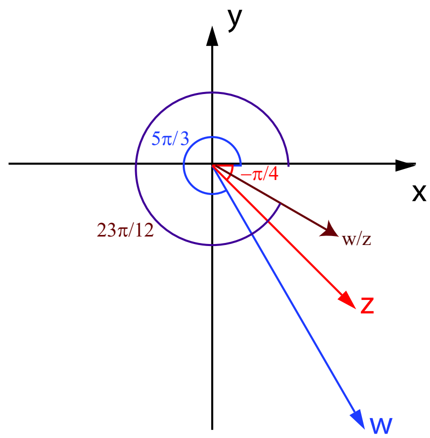
$$\frac{5\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{20\pi + 3\pi}{12} = \frac{23\pi}{12}.$$

3. The terminal side of an angle of $\dfrac{23\pi}{12} = 2\pi - \dfrac{\pi}{12}$ radians is in the fourth quadrant.

4. We know the magnitude and argument of wz, so the polar form of wz is

$$\frac{w}{z} = \frac{3}{2}\left[\cos\left(\frac{23\pi}{12}\right) + \sin\left(\frac{23\pi}{12}\right)\right].$$

5. Following is a picture of w, z, and wz that illustrates the action of the complex product.



Section 5.3

Progress Check 5.10

In polar form,

$$1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\right).$$

So

$$\begin{aligned}(1-i)^{10} &= \left(\sqrt{2}\right)^{10}\left(\cos\left(-\frac{10\pi}{4}\right) + \sin\left(-\frac{10\pi}{4}\right)\right)\\ &= 32\left(\cos\left(-\frac{5\pi}{2}\right) + \sin\left(-\frac{5\pi}{2}\right)\right)\\ &= 32(0-i)\\ &= -32i.\end{aligned}$$

Progress Check 5.13

1. We find the solutions to the equation $z^4 = 1$. Let $\omega = \cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$. Then

- $\omega^0 = 1$,
- $\omega = i$,
- $\omega^2 = \cos\left(\frac{2\pi}{2}\right) + i\sin\left(\frac{2\pi}{2}\right) = -1$

- $\omega^3 = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = -i$.

So the four fourth roots of unity are 1, i, -1, and $-i$.

2. We find the solutions to the equation $z^6 = 1$. Let $\omega = \cos\left(\frac{2\pi}{6}\right) + i \sin\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$. Then

- $\omega^0 = 1$,
- $\omega = \frac{1}{2} + \sqrt{3}2i$,
- $\omega^2 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \sqrt{3}2i$,
- $\omega^3 = \cos\left(\frac{3\pi}{3}\right) + i \sin\left(\frac{3\pi}{3}\right) = -1$,
- $\omega^4 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \sqrt{3}2i$,
- $\omega^5 = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \sqrt{3}2i$.

So the six fifth roots of unity are 1, $\frac{1}{2} + \sqrt{3}2i$, $-\frac{1}{2} + \sqrt{3}2i$, -1, $-\frac{1}{2} - \sqrt{3}2i$, and $\frac{1}{2} - \sqrt{3}2i$.

Progress Check 5.15

Since $-256 = 256\left[\cos(\pi) + i \sin(\pi)\right]$ we see that the fourth roots of -256 are

$$\begin{aligned} x_0 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(0)}{4}\right) + i \sin\left(\frac{\pi + 2\pi(0)}{4}\right)\right] \\ &= 4\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \\ &= 4\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] \\ &= 2\sqrt{2} + 2i\sqrt{2}, \end{aligned}$$

$$\begin{aligned} x_1 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(1)}{4}\right) + i \sin\left(\frac{\pi + 2\pi(1)}{4}\right)\right] \\ &= 4\cos\left(\frac{3\pi}{4}\right) + i \sin\left(3\frac{\pi}{4}\right) \\ &= 4\left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] \\ &= -2\sqrt{2} + 2i\sqrt{2}, \end{aligned}$$

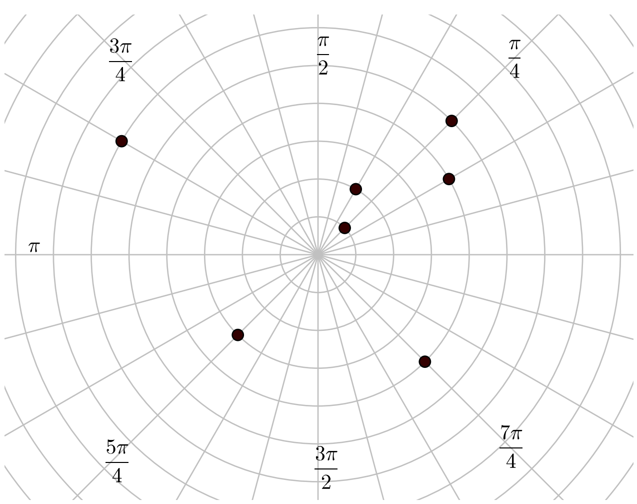
$$
\begin{aligned}
x_2 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi+2\pi(2)}{4}\right)+i\,\sin\left(\frac{\pi+2\pi(2)}{4}\right)\right]\\
&= 4\cos\left(\frac{5\pi}{4}\right)+i\,\sin\left(5\frac{\pi}{4}\right)\\
&= 4\left[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right]\\
&= -2\sqrt{2}-2i\sqrt{2},
\end{aligned}
$$

and

$$
\begin{aligned}
x_3 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi+2\pi(3)}{4}\right)+i\,\sin\left(\frac{\pi+2\pi(3)}{4}\right)\right]\\
&= 4\cos\left(\frac{7\pi}{4}\right)+i\,\sin\left(7\frac{\pi}{4}\right)\\
&= 4\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right]\\
&= 2\sqrt{2}-2i\sqrt{2}.
\end{aligned}
$$

Section 5.4

Progress Check 5.16



Progress Check 5.17

The left column shows some sets of polar coordinates with a positive value for r and the right column shows some sets of polar coordinates with a negative value for r.

$$\begin{array}{cc} (3, 470°) & (-3, 290°) \\ (3, 830°) & (-3, 650°) \\ (3, -250°) & (-3, -70°) \\ (3, -510°) & (-3, -430°) \\ (3, 1190°) & (-3, 1010°) \end{array}$$

Progress Check 5.18

For each point, we use the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$. In each of these cases, we can determine the exact values for x and y.

	Polar Coordinates	Rectangular Coordinates
1.	$\left(3, \frac{\pi}{3}\right)$	$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$
2.	$\left(5, \frac{11\pi}{6}\right)$	$\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$
3.	$\left(-5, \frac{3\pi}{4}\right)$	$\left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$

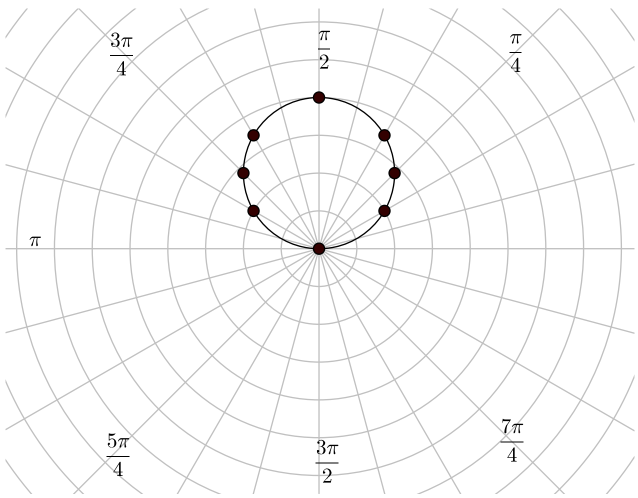
Progress Check 5.20

1. For the point $(6, 6\sqrt{3})$, $r^2 = 6^2 + \left(6\sqrt{3}\right)^2 = 144$ and so $r = 12$. Since the point is in the first quadrant, we can use $\tan(\theta) = \sqrt{3}$ or $\cos(\theta) = \frac{1}{2}$ to conclude that $\theta = \frac{\pi}{3}$. So the polar coordinates are $\left(12, \frac{\pi}{3}\right)$.

2. For the point $(0, -4)$, $r^2 = 0^2 + (-4)^2 = 16$ and so $r = 4$. Since the point is on the y-axis, we can use $\cos(\theta) = 0$ and $\sin(\theta) - 1$ to conclude that $\theta = \frac{3\pi}{2}$. So the polar coordinates are $\left(4, \frac{3\pi}{2}\right)$.

3. For the point $(-4, 5\sqrt{3})$, $r^2 = (-4)^2 + 5^2 = 41$ and so $r = \sqrt{41}$. Since the point is in the second quadrant, we can use $\tan(\theta) = -1.25$ to conclude that the reference angle is $\hat{\theta} = tan^{-1}(-1.25)$. We cannot determine an exact value for θ and so we can say that the polar coordinates are $\left(\sqrt{41}, \pi - \tan^{-1}(1.25)\right)$. We can also approximate the angle and see that the approximate polar coordinates are $\left(\sqrt{41}, 2.24554\right)$. Note: There are other ways to write the angle θ. It is also true that $\theta = \pi - \cos^{-1}\left(\dfrac{4}{\sqrt{21}}\right) = \cos^{-1}\left(\dfrac{-4}{\sqrt{21}}\right)$.

Progress Check 5.21



Progress Check 5.22

1. $r^2 = 4r\sin(\theta)$.

2. $x^2 + y^2 = 4y$.

Progress Check 5.23

$$r^2 = 6r\sin(\theta) \qquad x^2 - 6x + 9 + y^2 = 9$$
$$x^2 + y^2 = 6x \qquad (x-3)^2 + y^2 = 3^2$$

So the graph of $r = 3\cos(\theta)$ is a circle with radius 3 and center at $(3, 0)$.

Appendix B

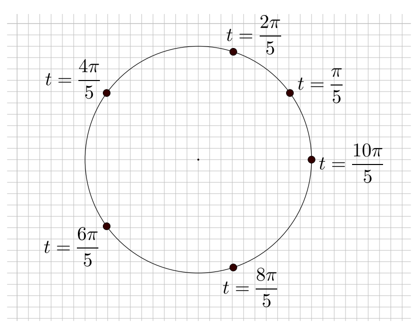
Answers and Hints for Selected Exercises

Section 1.1

1. **(b)**

t	point
1	$(0.54, 0.84)$
5	$(0.28, -0.96)$
9	$(-0.91, 0.41)$

2.



4.

	(a)	**(b)**	**(d)**	**(i)**	**(j)**	**(l)**	**(m)**
t	$\frac{7\pi}{4}$	$-\frac{7\pi}{4}$	$-\frac{3\pi}{5}$	2.5	-2.5	$3+2\pi$	$3-\pi$
Quadrant	IV	I	III	II	III	II	IV

5. **(a)** We substitute $x = \dfrac{1}{3}$ into the equation $x^2 + y^2 = 1$. Solving for y, we obtain $y = \pm\dfrac{\sqrt{8}}{3}$. So the points are $\left(\dfrac{1}{3}, \dfrac{\sqrt{8}}{3}\right)$ and $\left(\dfrac{1}{3}, -\dfrac{\sqrt{8}}{3}\right)$.

(b) We substitute $y = -\dfrac{1}{2}$ into the equation $x^2 + y^2 = 1$. Solving for x, we obtain $x = \pm\dfrac{\sqrt{3}}{2}$. So the points are $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ and $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.

Section 1.2

1. **(a)** For a real number t, the value of $\cos(t)$ is defined to be the ___x___-coordinate of the ___terminal___ point of an arc t whose initial point is ___$(1, 0)$___ on the ___unit circle___ whose equation is $x^2 + y^2 = 1$.

(b) The domain of the cosine function is ___the set of all real numbers___.

(c) The maximum value of $\cos(t)$ is ___1___ and this occurs at $t =$ ___0___ for $0 \leq t < 2\pi$. The minimum value of $\cos(t)$ is ___-1___ and this occurs at $t =$ ___π___ for $0 \leq t < 2\pi$.

(d) The range of the cosine function is ___the closed interval $[-1, 1]$___.

4. **(a)** $\cos(t) = \dfrac{4}{5}$ or $\cos(t) = -\dfrac{4}{5}$.

(c) $\sin(t) = -\dfrac{\sqrt{5}}{3}$.

5. **(a)** $0 < \cos^2(t) < \dfrac{1}{9}$.

(b) $-\dfrac{1}{9} < -\cos^2(t) < 0$ and so $\dfrac{8}{9} < 1 - \cos^2(t) < 1$

(c) $\dfrac{8}{9} < \sin^2(t) < 1$

(d) $\dfrac{\sqrt{8}}{3} < \sin(t) < 1$

Section 1.3

1. **(a)** $\frac{1}{12}\pi \approx 0.2618$

(b) $\frac{29}{90}\pi \approx 1.0123$

(e) $-\frac{2}{9}\pi \approx -0.6981$

2. **(a)** $67.5°$

(b) $231.4286°$

(d) $57.2958°$

4. **(a)** $\cos(10°) \approx 0.9848$, $\sin(10°) \approx 0.1736$

(d) $\cos(-10°) \approx 0.9848$, $\sin(-10°) \approx -0.1736$

Section 1.4

1. **(a)** The arc length is 4π feet, which is equal to $\frac{1}{3}$ of the circumference of the circle.

(b) The arc length is 200 miles.

(c) The arc length is 26π meters.

(d) The arc length is $\frac{1520}{180}\pi$ feet ≈ 26.53 feet.

2. **(a)** $\theta = \frac{3\pi}{5}$ radians.

(b) $\theta = \frac{18}{5}$ radians = 3.6 radians.

3. **(a)** $\theta = 108°$.

(b) $\theta = \left(\frac{648}{\pi}\right)^{\circ} \approx 206.26°$.

5. Earth travels through an angle of $\frac{2\pi}{365.25}$ radians in one day. Earth travels a distance of about 1.599 million miles in one day.

8. **(b)** $v = 720\pi \frac{\text{in}}{\text{min}} \approx 2261.95 \frac{\text{in}}{\text{min}}$.

9. **(b)** $v = 3600\pi \frac{\text{cm}}{\text{min}} \approx 11309.73 \frac{\text{cm}}{\text{min}}$.

Section 1.5

1. **(a)** $t = \dfrac{\pi}{3}$, $\cos(t) = \dfrac{1}{2}$, $\sin(t) = \dfrac{\sqrt{3}}{2}$.

(b) $t = \dfrac{\pi}{2}$, $\cos(t) = 0$, $\sin(t) = 1$.

(c) $t = \dfrac{\pi}{4}$, $\cos(t) = \dfrac{\sqrt{2}}{2}$, $\sin(t) = \dfrac{\sqrt{2}}{2}$.

(d) $t = \dfrac{\pi}{6}$, $\cos(t) = \dfrac{\sqrt{3}}{2}$, $\sin(t) = \dfrac{1}{2}$.

2. **(a)** $\cos^2\left(\dfrac{\pi}{6}\right) = \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{3}{4}$.

(b) $2\sin^2\left(\dfrac{\pi}{4}\right) + \cos(\pi) = 0$.

3. **(a)** The reference arc is $\dfrac{\pi}{3}$.

(b) The reference arc is $\dfrac{3\pi}{8}$.

(d) The reference arc is $\dfrac{\pi}{3}$.

4. **(a)** The reference arc is $\dfrac{\pi}{6}$; $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$; $\sin\left(\dfrac{5\pi}{6}\right) = -\dfrac{1}{2}$.

(d) The reference arc is $\dfrac{\pi}{3}$; $\cos\left(-\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$; $\sin\left(-\dfrac{2\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$.

6. **(a)** $\cos(t) = \dfrac{\sqrt{24}}{5}$.

(d) $\sin(\pi + t) = -\dfrac{1}{5}$.

Section 1.6

1.

t	$\cot(t)$	$\sec(t)$	$\csc(t)$
0	undefined	1	undefined
$\frac{\pi}{6}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
$\frac{\pi}{4}$	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$\frac{\pi}{2}$	0	undefined	1

3. **(a)** The terminal point is in the fourth quadrant.

(b) The terminal point is in the third quadrant.

4. $\cos(t) = -\dfrac{\sqrt{8}}{3}$ $\quad$ $\tan(t) = -\dfrac{1}{\sqrt{8}}$ $\quad$ $\sec(t) = -\dfrac{3}{\sqrt{8}}$

$\csc(t) = 3$ $\quad$ $\cot(t) = -\sqrt{8}$

8. **(a)** $t = \dfrac{5\pi}{4}$ $\quad$ **(b)** $t = \dfrac{\pi}{2}$

Section 2.1

1. **(a)** $C(\pi, -1)$ $\quad$ $R(\pi, 0)$

(b) $B\left(\dfrac{\pi}{3}, \dfrac{1}{2}\right)$ $\quad$ $Q\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$

2. **(a)** $y = 3\sin(x)$ $\quad$ **(b)** $y = 2\cos(x)$

3. **(a)** t-intercepts: $-2\pi, -\pi, 0, \pi, 2\pi$ $\quad$ y-intercept: $(0, 0)$

The maximum value is 1. Maximum value occurs at the points $\left(-\dfrac{3\pi}{2}, 1\right)$ and $\left(\dfrac{\pi}{2}, 1\right)$.

The minimum value is -1. Minimum value occurs at the points $\left(-\dfrac{\pi}{2}, -1\right)$ and $\left(\dfrac{3\pi}{2}, -1\right)$.

(b) t-intercepts: $-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ y-intercept: $(0, 2)$
The maximum value is 2. Maximum value occurs at the points $(0, 2)$ and $(2\pi, 2)$.
The minimum value is -2. Minimum value occurs at the points $(-\pi, -2)$, $(\pi, -2)$, and $(3\pi, -2)$.

Section 2.2

1. **(a)** $y = 2\sin(\pi x)$. The amplitude is 2; the period is 2; the phase shift is 0; and the vertical shift is 0.

$A(0, 0)$ $B\left(\dfrac{1}{2}, 2\right)$ $C(1, 0)$
$E\left(\dfrac{3}{2}, -2\right)$ $F(2, 0)$ $G\left(\dfrac{5}{2}, 2\right)$

(c) $y = 3\sin\left(x - \dfrac{\pi}{4}\right)$. The amplitude is 3; the period is 2π; the phase shift is $\dfrac{\pi}{4}$; and the vertical shift is 0.

$A\left(\dfrac{\pi}{4}, 0\right)$ $B\left(\dfrac{3\pi}{4}, 3\right)$ $C\left(\dfrac{5\pi}{4}, 0\right)$
$E\left(\dfrac{7\pi}{4}, -3\right)$ $F\left(\dfrac{9\pi}{4}, 0\right)$ $G\left(\dfrac{11\pi}{4}, 3\right)$

(g) $y = 4\sin\left(2\left(x - \dfrac{\pi}{4}\right)\right) + 1$. The amplitude is 4; the period is π; the phase shift is $\dfrac{\pi}{4}$; and the vertical shift is 1.

$A\left(\dfrac{\pi}{4}, 1\right)$ $B\left(\dfrac{\pi}{2}, 5\right)$ $C\left(\dfrac{3\pi}{4}, 1\right)$
$E\,(\pi, -3)$ $F\left(\dfrac{5\pi}{4}, 1\right)$ $G\left(\dfrac{3\pi}{2}, 5\right)$

2. **(a)** The amplitude is 2; the period is $\dfrac{2\pi}{3}$; and there is no vertical shift.

- For $y = A\sin(B(x-C)) + D$, there is no phase shift and so $C = 0$. So
$$y = 2\sin(3x).$$
- For $y = A\cos(B(x-C)) + D$, the phase shift is $\dfrac{\pi}{6}$ and so $C = \dfrac{\pi}{6}$. So
$$y = 2\cos\left(3\left(x - \frac{\pi}{6}\right)\right).$$

(d) The amplitude is 8; the period is 2; and the vertical shift is 1.

- For $y = A\sin(B(x-C)) + D$, the phase shift is $-\dfrac{1}{6}$ and so $C = -\dfrac{1}{6}$. So
$$y = 8\sin\left(\pi\left(x + \frac{1}{6}\right)\right).$$
- For $y = A\cos(B(x-C)) + D$, the phase shift is $\dfrac{1}{3}$ and so $C = \dfrac{1}{3}$. So
$$y = 8\cos\left(\pi\left(x - \frac{1}{3}\right)\right) + 1.$$

Section 2.3

1. **(a)** We write $y = 4\sin\left(\pi x - \dfrac{\pi}{8}\right) = 4\sin\left(\pi\left(x - \dfrac{1}{8}\right)\right)$. So the amplitude is 4, the period is 2, the phase shift is $\dfrac{1}{8}$, and there is no vertical shift.

- Some high points on the graph: $\left(\dfrac{5}{8}, 4\right)$, $\left(\dfrac{21}{8}, 4\right)$.
- Some low points on the graph: $\left(\dfrac{13}{8}, -4\right)$, $\left(\dfrac{29}{8}, -4\right)$.
- Graph crosses the center line at: $\left(\dfrac{1}{8}, 0\right)$, $\left(\dfrac{9}{8}, 0\right)$, $\left(\dfrac{17}{8}, 0\right)$.

(b) We write $y = 5\cos\left(4x + \dfrac{\pi}{2}\right) + 2 = 5\cos\left(4\left(x + \dfrac{\pi}{8}\right)\right) + 2$. So the amplitude is 5, the period is $\dfrac{\pi}{2}$, the phase shift is $-\dfrac{\pi}{8}$, and the vertical shift is 2.

- Some high points on the graph: $\left(-\frac{\pi}{8}, 7\right)$, $\left(\frac{3\pi}{8}, 7\right)$.
- Some low points on the graph: $\left(\frac{\pi}{8}, -3\right)$, $\left(\frac{5\pi}{8}, -3\right)$.
- Graph crosses the center line at: $(0, 2)$, $\left(\frac{\pi}{4}, 2\right)$, $\left(\frac{\pi}{2}, 2\right)$.

2. **(b)** The maximum value is 150 ml, and the minimum value is 81 ml. So we can use $A = \dfrac{150 - 81}{2} = 34.5$ and $D = \dfrac{150 + 81}{2} = 115.5$.

(c) The period is $\dfrac{1}{75}$ min.

Section 2.4

1. **(a)** $\tan(t + 2\pi) = \dfrac{\sin(t + 2\pi)}{\cos(t + 2\pi)} = \dfrac{\sin(t)}{\cos(t)} = \tan(t)$.

3. **(a)** $\csc(-t) = \dfrac{1}{\sin(-t)} = \dfrac{1}{-\sin(t)} = -\dfrac{1}{\sin(t)} = -\csc(t)$.

Section 2.5

1. **(a)** $t = \arcsin\left(\frac{\sqrt{2}}{2}\right)$ means $\sin(t) = \frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, we see that $t = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

(b) $t = \arcsin\left(-\frac{\sqrt{2}}{2}\right)$ means $\sin(t) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Since $\sin\left(\frac{-\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, we see that $t = \arcsin\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

(d) $t = \arccos\left(-\frac{\sqrt{2}}{2}\right)$ means $\cos(t) = -\frac{\sqrt{2}}{2}$ and $0 \le t \le \pi$. Since $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, we see that $t = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(f) $y = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ means $\tan(y) = -\frac{\sqrt{3}}{3}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Since $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$, we see that $y = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$.

(h) $t = \arctan(0) = 0$.

(j) $y = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

2. **(a)** $\sin\left(\sin^{-1}(1)\right) = \sin\left(\frac{\pi}{2}\right) = 1$

(b) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(e) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(f) $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(i) $\arctan\left(\tan\left(\frac{3\pi}{4}\right)\right) = \arctan(-1) = -\frac{\pi}{4}$

3. **(a)** Let $t = \arcsin\left(\frac{2}{5}\right)$. Then $\sin(t) = \frac{2}{5}$ and $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, and

$$\begin{aligned}\cos^2(t) + \sin(t)^2 &= 1\\ \cos^2(t) + \frac{4}{25} &= 1\\ \cos^2(t) &= \frac{21}{25}\end{aligned}$$

Since $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, we know that $\cos(t) \ge 0$. Hence, $\cos(t) = \frac{\sqrt{21}}{5}$ and $\cos\left(\arcsin\left(\frac{2}{5}\right)\right) = \frac{\sqrt{21}}{5}$.

(b) $\sin\left(\arccos\left(-\frac{2}{3}\right)\right) = \frac{\sqrt{5}}{3}$.

(c) $\tan\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{1}{\sqrt{8}}$.

Section 2.6

1. **(a)** $x = 0.848 + k(2\pi)$ or $x = 2.294 + k(2\pi)$, where k is an integer.

(d) $x = -0.848 + k(2\pi)$ or $x = -2.294 + k(2\pi)$, where k is an integer.

2. **(a)** $x = \sin^{-1}(0.75) + k(2\pi)$ or $x = \left(\pi - \sin^{-1}(0.75)\right) + k(2\pi)$, where k is an integer.

(d) $x = \arcsin(-0.75) + k(2\pi)$ or $x = (\pi - \arcsin(-0.75)) + k(2\pi)$, where k is an integer.

3. **(a)** $x = \sin^{-1}(0.4) + k(2\pi)$ or $x = \left(\pi - \sin^{-1}(0.4)\right) + k(2\pi)$, where k is an integer.

(b) $x = \cos^{-1)}\left(\dfrac{4}{5}\right) + k(2\pi)$ or $x = -\cos^{-1)}\left(\dfrac{4}{5}\right) + k(2\pi)$, where k is an integer.

4. **(a)** The period for the trigonometric function is π. We first solve the equation $4\sin(t) = 3$ with $-\pi \leq t \leq \pi$ and obtain $t = \sin^{-1}(0.75) + k(2\pi)$ or $t = \left(\pi - \sin^{-1}(0.75)\right) + k(2\pi)$. We then use the substitution $t = 2x$ to obtain

$x = \dfrac{1}{2}\sin^{-1}(0.75) + k(\pi)$ or $x = \dfrac{1}{2}\left(\pi - \sin^{-1}(0.75)\right) + k(\pi)$, where k is an integer.

(d) The period for the trigonometric function is 2. We first solve the equation $\sin(t) = 0.2$ with $-\pi \leq t \leq \pi$ and obtain $t = \sin^{-1}(0.2) + k(2\pi)$ or $t = \left(\pi - \sin^{-1}(0.2)\right) + k(2\pi)$. We now use the substitution $t = \pi x - \dfrac{\pi}{4}$ to obtain

$x = \left(\dfrac{1}{\pi}\sin^{-1}(0.2) + \dfrac{1}{4}\right) + 2k$ or $x = \left(-\dfrac{1}{\pi}\sin^{-1}(0.2) + \dfrac{5}{4}\right) + 2k$, where k is an integer.

Section 3.1

1. **(a)** We see that $r = \sqrt{3^2 + 3^2} = \sqrt{18}$. So

$$\cos(\theta) = \frac{3}{\sqrt{18}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \sin(\theta) = \frac{3}{\sqrt{18}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\tan(\theta) = \frac{3}{3} = 1 \qquad \cot(\theta) = \frac{3}{3} = 1$$
$$\sec(\theta) = \frac{\sqrt{18}}{3} = \sqrt{2} \qquad \csc(\theta) = \frac{\sqrt{18}}{3} = \sqrt{2}$$

(b) We see that $r = \sqrt{5^2 + 8^2} = \sqrt{89}$. So

$$\cos(\theta) = \frac{5}{\sqrt{89}} \qquad \sin(\theta) = \frac{8}{\sqrt{89}}$$
$$\tan(\theta) = \frac{8}{5} \qquad \cot(\theta) = \frac{5}{8} = 1$$
$$\sec(\theta) = \frac{\sqrt{89}}{5} \qquad \csc(\theta) = \frac{\sqrt{89}}{8}$$

(e) We see that $r = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$. So

$$\cos(\theta) = \frac{-1}{\sqrt{17}} \qquad \sin(\theta) = \frac{-4}{\sqrt{17}}$$
$$\tan(\theta) = 4 \qquad \cot(\theta) = \frac{1}{4}$$
$$\sec(\theta) = -\sqrt{17} - \frac{\sqrt{26}}{4} \qquad \csc(\theta) = -\frac{\sqrt{17}}{4}$$

2. **(b)** We first use the Pythagorean Identity and obtain $\sin^2(\beta) = \frac{5}{9}$. Since the terminal side of β is in the second quadrant, $\sin(\beta) = \frac{\sqrt{5}}{3}$. In addition,

$$\tan(\beta) = -\frac{\sqrt{5}}{2} \qquad \cot(\beta) = -\frac{2}{\sqrt{5}}$$
$$\sec(\beta) = -\frac{3}{2} \qquad \csc(\beta) = \frac{3}{\sqrt{5}}$$

3. **(c)** Since the terminal side of θ is in the second quadrant, θ is not the inverse sine of $\frac{2}{3}$. So we let $\alpha = \arcsin\left(\frac{2}{3}\right)$. Using α as the reference angle, we then see that

$$\theta = 180° - \arcsin\left(\frac{2}{3}\right) \approx 138.190°.$$

(e) $\theta = \arccos\left(-\frac{1}{4}\right) \approx 104.478°$, or use $\alpha = \arccos\left(\frac{1}{4}\right)$ for the reference angle.

$$\theta = 180° - \arccos\left(\frac{1}{4}\right) \approx 104.478°.$$

4. **(c)** $\theta = \pi - \arcsin\left(\frac{2}{3}\right) \approx 2.142.$

(e) $\theta = \pi - \arccos\left(\frac{1}{4}\right) \approx 1.823.$

5. **(b)** Let $\theta = \cos^{-1}\left(\frac{2}{3}\right)$. Then $\cos(\theta) = \frac{2}{3}$ and $0 \leq \theta \leq \pi$. So $\sin(\theta) > 0$ and $\sin^2(\theta) = 1 - \cos^2(\theta) = \frac{5}{9}$. So

$$\begin{aligned}
\tan\left(\cos^{-1}\left(\frac{2}{3}\right)\right) &= \tan(\theta) \\
&= \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} \\
&= \frac{\sqrt{5}}{2}
\end{aligned}$$

Using a calculator, we obtain

$$\tan\left(\cos^{-1}\left(\frac{2}{3}\right)\right) \approx 1.11803 \quad \text{and} \quad \frac{\sqrt{5}}{2} \approx 1.11803.$$

Section 3.2

1. (a) $x = 6\tan(47°) \approx 6.434$.
   (b) $x = 3.1\cos(67°) \approx 1.211$.
   (c) $x = \tan^{-1}\left(\dfrac{7}{4.9}\right) \approx 55.008°$.
   (d) $x = \sin^{-1}\left(\dfrac{7}{9.5}\right) \approx 47.463°$.

4. The other acute angle is $64°48'$.
   - The side opposite the $27°12'$ angle is $4\tan(27°12') \approx 2.056$ feet.
   - The hypotenuse is $\dfrac{4}{\cos(27°12')} \approx 4.497$ feet.

   Note that the Pythagorean Theorem can be used to check the results by showing that $4^2 + 2.056^2 \approx 4.497^2$. The check will not be exact because the 2.056 and 4.497 are approximations of the exact values.

7. We first note that $\theta = 40°$. We use the following two equations to determine x.

$$\tan(\alpha) = \frac{h}{c+x} \qquad\qquad \tan(\theta) = \frac{h}{x}$$

   Substituting $h = x\tan(\theta)$ into the first equation and solving for x gives

$$x = \frac{c\tan(\alpha)}{\tan(\theta) - \tan(\alpha)} \approx 8.190.$$

   We can then use right triangles to obtain $h \approx 6.872$ ft, $a \approx 10.691$ ft, and $b \approx 17.588$ ft.

Section 3.3

1. The third angle is $65°$. The side opposite the $42°$ angle is 4.548 feet long. The side opposite the $65°$ angle is 6.160 feet long.

3. There are two triangles that satisfy these conditions. The sine of the angle opposite the 5 inch side is approximately 0.9717997.

5. The angle opposite the 9 foot long side is $95.739°$. The angle opposite the 7 foot long side is $50.704°$. The angle opposite the 5 foot long side is $33.557°$.

Section 3.4

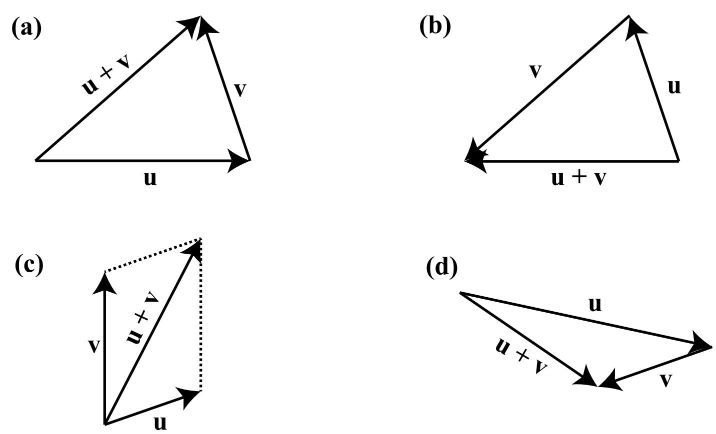
1. The ski lift is about 1887.50 feet long.
2. **(a)** The boat is about 67.8 miles from Chicago.

   (b) $\gamma \approx 142.4°$. So the boat should turn through an angle of about $180° - 142.4° = 37.6°$.

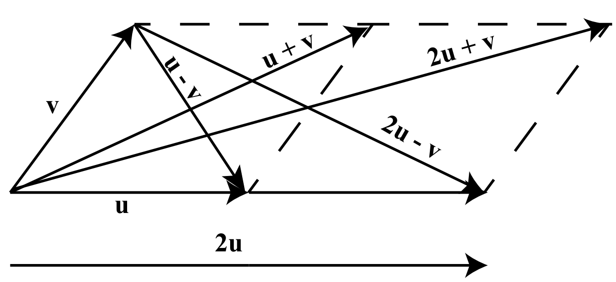
   (c) The direct trip from Muskegon to Chicago would take $\dfrac{121}{15}$ hours or about 8.07 hours. By going off-course, the trip now will take $\dfrac{127.8}{15}$ hours or about 8.52 hours.

Section 3.5

1.



2.



3. The angle between the vectors **a** and **a** + **b** is approximately 9.075°. In addition, $|\mathbf{b}| \approx 4.416$.

Section 3.6

1. **(a)** $|\mathbf{v}| = \sqrt{34}$. The direction angle is approximately 59.036°.

   (b) $|\mathbf{w}| = \sqrt{45}$. The direction angle is approximately 116.565°.

2. **(a)** $\mathbf{v} = 12\cos(50°) + 12\sin(50°) \approx 7.713\mathbf{i} + 9.193\mathbf{j}$.

   (b) $\mathbf{u} = \sqrt{20}\cos(125°) + \sqrt{20}\sin(125°) \approx -2.565\mathbf{i} + 3.663\mathbf{j}$.

3. **(a)** $5\mathbf{u} - \mathbf{v} = 11\mathbf{i} + 10\mathbf{j}$.

   (c) $\mathbf{u} + \mathbf{v} + \mathbf{w} = 5\mathbf{i} + 6\mathbf{j}$.

4. **(a)** $\mathbf{v}\cdot\mathbf{w} = -4$.

   (b) $\mathbf{a}\cdot\mathbf{b} = 9\sqrt{3}$.

5. **(a)** The angle between **v** and **w** is $\cos^{-1}\left(\frac{-4}{\sqrt{29}\sqrt{13}}\right) \approx 101.89°$.

6. **(a)** $\mathbf{proj}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-4}{29}(2\mathbf{i} + 5\mathbf{j}) = -\frac{8}{29}\mathbf{i} - \frac{20}{29}\mathbf{j}$.

   $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{w} = \mathbf{w} - \mathbf{proj}_{\mathbf{v}}\mathbf{w} = \frac{95}{29}\mathbf{i} - \frac{38}{29}\mathbf{j}$

Section 4.1

1. **(a)**

$$\begin{aligned}\cos(x)\tan(x) &= \cos(x)\frac{\sin(x)}{\cos(x)}\\ &= \sin(x)\end{aligned}$$

(b)

$$\begin{aligned}\frac{\cot(x)}{(x)} &= \frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}}\\ &= \frac{\cos(x)}{\sin(x)} \times \frac{\sin(x)}{1}\\ &= \cos(x)\end{aligned}$$

(e) A graph will show that this is not an identity. In particular, we see that

$$\sec^2\left(\frac{\pi}{4}\right)+\csc^2\left(\frac{\pi}{4}\right)=\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2=4$$

Section 4.2

1. **(a)** $x=\dfrac{\pi}{6}+k(2\pi)$ or $x=\dfrac{5\pi}{6}+k(2\pi)$, where k is an integer.

   (b) $x=\dfrac{2\pi}{3}+k(2\pi)$ or $x=\dfrac{4\pi}{3}+k(2\pi)$, where k is an integer.

   (d) $x=\cos^{-1}\left(\dfrac{3}{4}\right)+k(2\pi)$ or $x=\cos^{-1}\left(-\dfrac{3}{4}\right)+k(2\pi)$, where k is an integer.

   (f) $x=k\pi$, where k is an integer.

2. $\theta=\sin^{-1}\left(\dfrac{2}{3}\right)\approx 41.81^\circ$ is one solution of the equation $\sin(\theta)+\dfrac{1}{3}=1$ with $0\le\theta\le 360^\circ$. There is another solution (in the second quadrant) for this equation with $0\le\theta\le 360^\circ$.

Section 4.3

1. **(a)** $\cos\left(-10^\circ-35^\circ\right)=\cos\left(-45^\circ\right)=\dfrac{\sqrt{2}}{2}$.

   (b) $\cos\left(\dfrac{7\pi}{9}+\dfrac{2\pi}{9}\right)=\cos(\pi)=-1$.

2. We first use the Pythagorean Identity to determine $\cos(A)$ and $\sin(B)$. From this, we get

$$\cos(A)=\frac{\sqrt{3}}{2}\quad\text{and}\quad\sin(B)=-\frac{\sqrt{7}}{4}.$$

(a)

$$\begin{aligned}\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\ &= \frac{\sqrt{3}}{2}\cdot\frac{3}{4} - \frac{1}{2}\cdot\left(-\frac{\sqrt{7}}{4}\right)\\ &= \frac{3\sqrt{3}+\sqrt{7}}{8}\end{aligned}$$

3. **(a)** $\cos(15°) = \cos(45° - 30°) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

(d) We can use $345° = 300° + 45°$ and first evaluate $\cos(345°)$. This gives $\cos(345°) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ and $\sec(345°) = \dfrac{4}{\sqrt{6}+\sqrt{2}}$. We could have also used the fact that $\cos(345°) = \cos(15°)$ and the result in part (a).

5. **(a)**

$$\begin{aligned}\cot\left(\frac{\pi}{2}-x\right) &= \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}\\ &= \frac{\sin(x)}{\cos(x)}\\ &= \tan(x)\end{aligned}$$

9. **(c)** It can be shown that $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = \sin(x)+\cos(x)$. So the graph of $f(x) = \sin(x)+\cos(x)$ has an amplitude of $\sqrt{2}$, a phase shift of $-\dfrac{\pi}{4}$, and a period of 2π.

Section 4.4

1. Use the Pythagorean Identity to obtain $\sin^2(\theta) = \dfrac{5}{9}$. Since $\sin(\theta) < 0$, we see that $\sin(\theta) = -\dfrac{\sqrt{5}}{3}$. Now use appropriate double angle identities to get

$$\sin(2\theta) = -\frac{4\sqrt{5}}{9} \qquad\qquad \cos(2\theta) = -\frac{1}{9}$$

Then use $\tan(2\theta) = \dfrac{\sin(2\theta)}{\cos(2\theta)} = 4\sqrt{5}$.

2. **(a)** $x = \dfrac{\pi}{4} + k\pi$, where k is an integer.

3. **(a)** This is an identity. Start with the left side of the equation and use $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$ and $\sin(2t) = 2\sin(t)\cos(t)$.

6. **(a)** $\sin(22.5°) = \sqrt{\dfrac{1 - \frac{\sqrt{2}}{2}}{2}} = \dfrac{1}{2}\sqrt{2 - \sqrt{2}}$.

(c) $\tan(22.5°) = \sqrt{\dfrac{2 - \sqrt{2}}{2 + \sqrt{2}}} = \sqrt{3 - 2\sqrt{2}}$.

(h) $\cos(195°) = -\sqrt{\dfrac{1 + \frac{\sqrt{3}}{2}}{2}} = -\dfrac{1}{2}\sqrt{2 + \sqrt{3}}$.

7. **(a)** $\sin\left(\dfrac{3\pi}{8}\right) = \sqrt{\dfrac{1 + \frac{\sqrt{2}}{2}}{2}} = \dfrac{1}{2}\sqrt{2 + \sqrt{2}}$.

(c) $\tan\left(\dfrac{3\pi}{8}\right) = \sqrt{\dfrac{2 + \sqrt{2}}{2 - \sqrt{2}}} = \sqrt{3 + 2\sqrt{2}}$.

(h) $\cos\left(\dfrac{11\pi}{12}\right) = -\sqrt{\dfrac{1 + \frac{\sqrt{3}}{2}}{2}} = -\dfrac{1}{2}\sqrt{2 + \sqrt{3}}$.

8. **(a)** We note that since $\dfrac{3\pi}{2} \leq x \leq 2\pi$, $\dfrac{3\pi}{4} \leq \dfrac{x}{2} \leq \pi$.

$$\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1 - \frac{2}{3}}{2}} = \frac{1}{\sqrt{6}}.$$

Section 4.5

1. **(a)** $\sin(37.5°)\cos(7.5°) = \dfrac{1}{2}[\sin(45°) + \sin(30°)] = \dfrac{\sqrt{2} + 1}{4}$

(e) $\cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right)\right] = \frac{2-\sqrt{3}}{4}$

2. **(a)** $\sin(50°) + \sin(10°) = 2\sin(30°)\cos(20°) = \cos(20°)$

(e) $\cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right) = 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

3. **(a)**

$$\sin(2x) + \sin(x) = 0$$
$$2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$$

So $\sin\left(\frac{3x}{2}\right) = 0$ or $\cos\left(\frac{x}{2}\right) = 0$. This gives

$$x = k\pi \quad \text{or} \quad x = \frac{2\pi}{3} + k(2\pi) \quad \text{or} \quad x = \frac{4\pi}{3} + k(2\pi),$$

where k is an integer.

Section 5.1

1. **(a)** $(4+i) + (3-3i) = 7-2i$

(b) $5(2-i) + i(3-2i) = 12-2i$

(c) $(4+2i)(5-3i) = 26-2i$

(d) $(2+3i)(1+i) + (4-3i) = 3+2i$

2. **(a)** $x = \frac{3}{2} + \frac{\sqrt{11}}{2}i$, $x = \frac{3}{2} - \frac{\sqrt{11}}{2}i$.

3. **(a)** $w + z = 8 - 2i$. **(b)** $w + z = -3 + 6i$.

4. **(a)** $\overline{z} = 5 + 2i$, $|z| = \sqrt{29}$, $z\overline{z} = 29$.

(b) $\overline{z} = -3i$, $|z| = 3$, $z\overline{z} = 9$.

5. **(a)** $\frac{5+i}{3+2i} = \frac{17}{13} - \frac{7}{13}i$. **(b)** $\frac{3+3i}{i} = 3 - 3i$.

Section 5.2

1. **(a)** $3 + 3i = \sqrt{18}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right)$

(e) $4\sqrt{3} + 4i = 8\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right)$

2. **(a)** $5\left(\cos\left(\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{2}\right)\right) = 5i$

(b) $2.5\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right) = 1.25\sqrt{2} + 1.25\sqrt{2}i$

3. **(a)** $wz = 10\left(\cos\left(\frac{6\pi}{12}\right) + i\,\sin\left(\frac{6\pi}{12}\right)\right) = 10i$

(b) $wz = 6.9\left(\cos\left(\frac{19\pi}{12}\right) + i\,\sin\left(\frac{19\pi}{12}\right)\right)$

4. **(a)** $\frac{w}{z} = \frac{5}{2}\left(\cos\left(\frac{-4\pi}{12}\right) + i\,\sin\left(\frac{-4\pi}{12}\right)\right) = \frac{5}{4} - \frac{5\sqrt{3}}{4}i$

(b) $\frac{w}{z} = \frac{23}{30}\left(\cos\left(\frac{-11\pi}{12}\right) + i\,\sin\left(\frac{-11\pi}{12}\right)\right)$

Section 5.3

1. **(a)** $(2+2i)^6 = \left[\sqrt{8}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right)\right]^6 = 512i$

(b) $\left(\sqrt{3}+i\right)^8 = \left[2\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right)\right]^8 = -128 - 128\sqrt{3}i$

2. **(a)** Write $16i = 16\left(\cos\left(\frac{\pi}{2}\right) + i\,\sin(\left(\frac{\pi}{2}\right)\right)$. The two square roots of $16i$ are

$$4\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right) = 2\sqrt{2} + 2i\sqrt{2}$$
$$4\left(\cos\left(\frac{5\pi}{4}\right) + i\,\sin\left(\frac{5\pi}{4}\right)\right) = -2\sqrt{2} - 2i\sqrt{2}$$

(c) The three cube roots of $5\left(\cos\left(\frac{3\pi}{4}\right)+i\,\sin\left(\frac{3\pi}{4}\right)\right)$ are

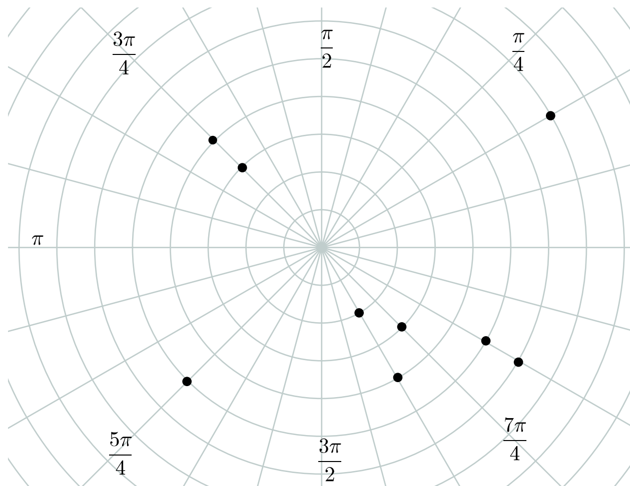
$$\sqrt[3]{5}\left(\cos\left(\frac{\pi}{4}\right)+i\,\sin\left(\frac{\pi}{4}\right)\right)=\sqrt[3]{5}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$\sqrt[3]{5}\left(\cos\left(\frac{11\pi}{12}\right)+i\,\sin\left(\frac{11\pi}{12}\right)\right)$$

$$\sqrt[3]{5}\left(\cos\left(\frac{19\pi}{12}\right)+i\,\sin\left(\frac{19\pi}{12}\right)\right)$$

Section 5.4

1.



2. **(a)** Some correct answers are: $(5, 390°)$, $(5, -330°)$, and $(-5, 210°)$.

(b) Some correct answers are: $(4, 460°)$, $(4, -260°)$, and $(-4, 280°)$.

3. **(a)** Some correct answers are: $\left(5, \dfrac{13\pi}{6}\right)$, $\left(5, -\dfrac{11\pi}{6}\right)$, and $\left(-5, \dfrac{7\pi}{6}\right)$.

(b) Some correct answers are: $\left(4, \dfrac{23\pi}{9}\right)$, $\left(4, -\dfrac{13\pi}{9}\right)$, and $\left(-4, \dfrac{14\pi}{9}\right)$.

4. **(a)** $(-5, 5\sqrt{3})$.

(c) $\left(\dfrac{5\sqrt{2}}{2}, \dfrac{5\sqrt{2}}{2}\right)$

5. **(a)** $\left(5, \dfrac{5\pi}{6}\right)$.

(b) $\left(\sqrt{34}, \tan^{-1}\left(\dfrac{5}{3}\right)\right) \approx \left(\sqrt{34}, 1.030\right)$

6. **(a)** $x^2 + y^2 = 25$

(b) $y = \dfrac{\sqrt{3}}{3}x$

(d) $x^2 + y^2 = \sqrt{x^2 + y^2} - y$

7. **(b)** $r\sin(\theta) = 4$ or $r = \dfrac{4}{\sin(\theta)}$

(e) $r\cos(\theta) + r\sin(\theta) = 4$ or $r = \dfrac{4}{\cos(\theta) + \sin(\theta)}$

Appendix C

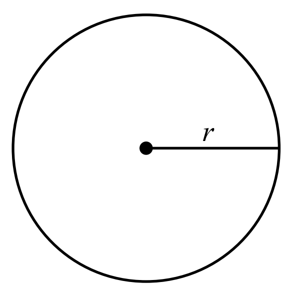
Some Geometric Facts about Triangles and Parallelograms

This appendix contains some formulas and results from geometry that are important in the study of trigonometry.

Circles

For a circle with radius r:

- **Circumference**: $C = 2\pi r$
- **Area**: $A = \pi r^2$



Triangles

- The sum of the measures of the three angles of a triangle is 180°.
- A triangle in which each angle has a measure of less that 90° is called an **acute triangle**.
- A triangle that has an angle whose measure is greater than 90° is called an **obtuse triangle**.
- A triangle that contains an angle whose measure is 90° is called a **right triangle**. The side of a right triangle that is opposite the right angle is called

the **hypotenuse**, and the other two sides are called the **legs**.

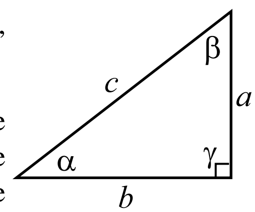
- An **isosceles triangle** is a triangle in which two sides of the triangle have equal length. In this case, the two angles across from the two sides of equal length have equal measure.

- An **equilateral triangle** is a triangle in which all three sides have the same length. Each angle of an equilateral triangle has a measure of 60°.

Right Triangles

- The sum of the measures of the two acute angles of a right triangle is 90°. In the diagram on the right, $\alpha + \beta = 90°$.

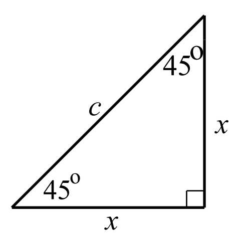
- **The Pythagorean Theorem**. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. In the diagram on the right, $c^2 = a^2 + b^2$.



Special Right Triangles

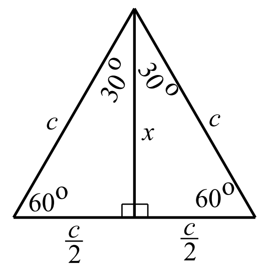
- **A right triangle in which both acute angles are 45°**. For this type of right triangle, the lengths of the two legs are equal. So if c is the length of the hypotenuse and x is the length of each of the legs, then by the Pythagorean Theorem, $c^2 = x^2 + x^2$. Solving this equation for x, we obtain

$$\begin{aligned} 2x^2 &= c^2 \\ x^2 &= \frac{c^2}{2} \\ x &= \sqrt{\frac{c^2}{2}} \\ x &= \frac{c}{\sqrt{2}} = \frac{\sqrt{2}}{2}c \end{aligned}$$



- **A right triangle with acute angles of 30° and 60°.**

We start with an equilateral triangle with sides of length c. By drawing an angle bisector at one of the vertices, we create two congruent right triangles with acute angles of 30° and 60°.

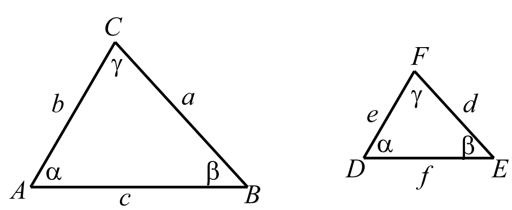


This means that the third side of each of these right triangles will have a length of $\dfrac{c}{2}$. If the length of the altitude is x, then using the Pythagorean Theorem, we obtain

$$\begin{aligned} c^2 &= x^2 + \left(\frac{c}{2}\right)^2 \\ x^2 &= c^2 - \frac{c^2}{4} \\ x^2 &= \frac{3c^2}{4} \\ x &= \sqrt{\frac{3c^2}{4}} = \frac{\sqrt{3}}{2}c \end{aligned}$$

Similar Triangles

Two triangles are **similar** if the three angles of one triangle are equal in measure to the three angles of the other triangle. The following diagram shows similar triangles $\triangle ABC$ and $\triangle DEF$. We write $\triangle ABC \sim \triangle DEF$.

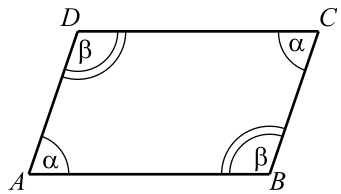


The sides of similar triangles do not have to have the same length but they will be proportional. Using the notation in the diagram, this means that

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Parallelograms

We use some properties of parallelograms in the study of vectors in Section 3.5. A **parallelogram** is a quadrilateral with two pairs of parallel sides. We will use the diagram on the right to describe some properties of parallelograms.



- Opposite sides are equal in length. In the diagram, this means that

$$AB = DC \text{ and } AD = BC.$$

- As shown in the diagram, opposite angles are equal. That is,

$$\angle DAB = \angle BCD \text{ and } \angle ABC = \angle CDA.$$

- The sum of two adjacent angles is 180°. In the diagram, this means that

$$\alpha + \beta = 180°.$$

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